

STRUCTURAL STABILITY OF COLUMNS AND PLATES

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Preface

The early books on stability of structures by S.P. Timoshenko, and F. Bleich treat a variety of problems normally encountered in engineering. However, the discussion in these excellent texts is limited to classical techniques and to metallic materials. In view of the rapid advancement in computer technology and the development of new techniques for solving engineering problems, the classical techniques have become purely of academic interest. Further, new materials such as fibre reinforced composites are increasingly being used because of the many advantages they have over the conventional metallic materials.

Today, the numerical techniques, for example, the finite element and finite difference techniques, are extensively applied for the solution of real-life problems. Accordingly, a large number of good books covering such techniques is available. However, each presents the applicability of only one or the other technique to structural stability problems. This text, on the other hand, lucidly discusses the structural applications of both the categories of techniques, i.e., numerical and classical, and, in the process, also compares their performance. It essentially consists of two 'sections'. In one, mainly the one-dimensional structures, viz., columns, beams, and frames, are studied and, in the other, the two-dimensional structures, namely, plates, are dealt with. In both the sections, the elastic and inelastic behaviour of the structures is analyzed. The emphasis throughout is on the illustration of the techniques through numerous worked-out numerical examples. There is a conscious effort to present the results in a nondimensional form so that these results are applicable for different materials, including composite materials.

The text is specially designed for the undergraduate/graduate in aeronautical, civil, and mechanical engineering. It would also be useful to the practising engineer.

I am indebted to my graduate students, M.K. Patra, P.J. Soni, and S.P. Joshi, for the help they extended in solving the numerical problems. Also, the financial assistance granted by the Quality Improvement Programme established by the Ministry of Education and Culture at the Indian Institute of Technology, Kanpur, for the preparation of the manuscript is gratefully acknowledged. Finally, I am extremely thankful to my wife, Leela, and sons, Garud and Raghuram, for the patience they showed while I was busy completing this effort.

N.G.R. Iyengar

1

Elastic Buckling of Columns

1.1 INTRODUCTION

The design of any modern structural system, for example, an aircraft, a rocket, a tall structure, and a complicated machine, depends on a large number of factors, namely, the economic aspects, material availability, response of each structure of the system to static and dynamic loads, temperature effects, and so on. The designer is interested in arriving at an optimum design, taking into consideration all these factors.

Structural analysis is normally concerned with the behaviour of a structure or the elements of a structure under the action of external loads. The two important questions usually asked when analyzing a structure are:

(i) What is the response of the structure when subjected to external mechanical loads and/or temperature changes? In other words, if the external loads are known, can we find the deformation pattern of and the internal stress distribution in the structure?

(ii) What is the nature of the response of the structure? In other words, is the equilibrium of the structure stable or unstable?

An understanding of (i) and (ii) is necessary for designing a safe structure.

In many instances, instability is not directly associated with the failure of the overall structure. For example, in an aircraft, if the skin wrinkles or a stringer locally buckles, the entire fuselage or wing does not fail. However, if a portion of the fuselage between two adjacent rings becomes unstable or a wing panel becomes unstable, the entire fuselage or wing fails catastrophically. Thus, stability also plays an important part in designing a structure.

A structure may have two kinds of failure, namely, (i) material failure and (ii) form failure. In material failure, the stresses in the structure exceed the specified safe limit, resulting in the formation of cracks which cause failure. In form failure, though the stresses may not exceed the safe value, the structure may not be able to maintain its original form. Here, the structure does not fail physically but may deform to some other shape due to intolerable external disturbance. Further, form failure depends on the geometry and loading of the structure. It occurs when the conditions of loading are such that compressive stresses get introduced. To understand the cause of this failure, we need to know not only the equilibrium of the structure but also the nature of the equilibrium. When the magnitude of the load on a structure is such that the equilibrium changes from stable to neutral, the load is called the *critical load* (or the limiting load at which the straight configuration of the structure ceases to be

stable). This phenomenon of change of equilibrium is called the *buckling of a structure*. The term “buckling load” is used synonymously with “critical load”. However, there is a subtle difference between the two terms. The latter defines the load, obtained theoretically, for an ideal structure, whereas the former is the load, obtained actually, for a real structure. The two may differ depending on the idealization.

Even a simple structural element under an axial compressive loading behaves in a complex manner. In structural design, all types of structural sections are used. These can be broadly classified as closed sections and open sections. The instability of these sections depends on (i) flexural rigidity EI , (ii) torsional rigidity GJ , (iii) position of shear centre with respect to the centre of gravity, (iv) symmetry of cross-section, and (v) material behaviour. In order to study the behaviour of a column and to investigate the effect of each of these parameters, it is advisable to start with a simple model and then improve it by making realistic assumptions. In so doing, the questions that arise are: Why does the column buckle? And how is the load to be estimated? These questions can be easily answered if we ascertain whether there is any equilibrium configuration other than the straight one. Such a probe could be conducted in several ways (depending on the method of generating the alternative configurations).

1.2 STABILITY CRITERIA

The techniques for generating alternative configurations can be broadly classified as static and dynamic. Such methods as use the equilibrium (Euler) approach, energy approach, and imperfection approach are called static. The method using the vibration approach is termed as dynamic. In what follows, we shall discuss each of these approaches.

1.2.1 Equilibrium Approach

The equilibrium approach deals with the equilibrium configuration of a perfect system. As we have already stated, the behaviour of a column under an axially compressive load is fairly complex. We shall therefore first study the behaviour of an idealized column, the Euler column.

The column shown in Fig. 1.1 is assumed to have a uniform cross-section throughout the

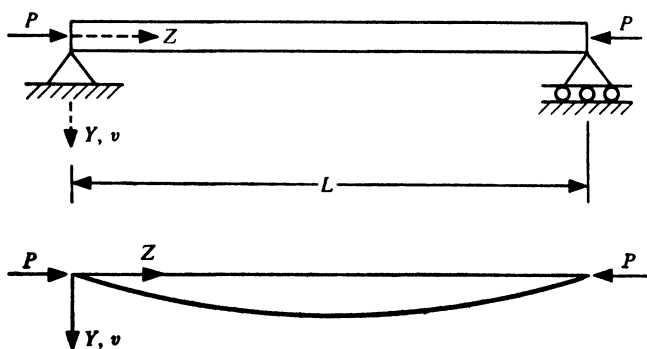


Fig. 1.1 Simply-Supported Column under Axial Load.

length and to be made of a homogeneous material. We shall derive the equilibrium equation for this column by assuming that

- (i) the column is axially loaded, i.e., the centre of gravity of loading passes through the centre of gravity of the cross-section,
- (ii) the material is linearly elastic,
- (iii) the deflections are small ($v \ll h$), and
- (iv) the ends are simply-supported,

where v is the transverse displacement and h is the thickness of the beam.

In view of assumption (iii), the moment-curvature relation becomes linear and can be written as

$$M = -EI \frac{d^2v}{dz^2}, \quad (1.1)$$

where M is the bending moment and I is the second moment of area. For a deformed shape to be in equilibrium, the internal resistive moment and the external moment must be equal. The external moment is Pv . Hence,

$$d^2v/dz^2 + k^2v = 0, \quad (1.2)$$

where $k^2 = P/(EI)$. Equation (1.2) is a linear homogeneous differential equation with constant coefficients. The most general solution for v can be written as

$$v = A \sin kz + B \cos kz. \quad (1.3)$$

The constants A and B are evaluated from the boundary conditions, which are (see Fig. 1.1)

$$EI \frac{d^2v}{dz^2} = 0 \quad (z = 0, L), \quad (1.4)$$

$$v = 0 \quad (z = 0, L). \quad (1.5)$$

Relation (1.4) is identically satisfied in view of Eq. (1.2) provided Eq. (1.5) is satisfied. Using relation (1.5), we get

$$B = 0, \quad (1.6a)$$

$$A \sin kL = 0. \quad (1.6b)$$

Relation (1.6b) can be satisfied in two possible ways, namely,

$$A = 0 \quad \text{or} \quad \sin kL = 0.$$

If $A = 0$, then we arrive at a trivial solution, that is, the straight configuration, which is known to be in equilibrium under any axial load. Hence, A cannot be zero. If $\sin kL = 0$, then

$$kL = n\pi, \quad (1.7)$$

where $n = 1, 2, 3 \dots$. The load P and the deflected shape are then given by

$$P = n^2 \pi^2 EI / L^2, \quad (1.8)$$

$$v = A \sin \frac{n\pi z}{L}. \quad (1.9)$$

Relation (1.8) gives the axial load P at which the column can be in equilibrium even when in a slightly bent form, as shown in Fig. 1.1. This load can have infinite discrete values. Each such value gives the critical load. The load P_1 corresponding to $n = 1$ is called the first critical load for $n = 2$, and so on.

The deflected shape or the mode shape of buckling is given by Eq. (1.9). However, here the amplitude of deflection is indeterminate since A can have any value. A problem yielding such a solution is governed by a linear homogeneous differential equation and homogeneous boundary conditions and is characterized as an eigenvalue problem. In Section 1.6, we shall show that this is true only when a small deformation is considered.

The value of P , obtained by setting $n = 1$, is

$$P_{cr} = \pi^2 EI / L^2. \quad (1.10)$$

This load is known as the Euler load. It is the smallest load at which a column ceases to be in stable equilibrium. Figure 1.2 shows the load versus displacement curve for a column. As can be seen, up to the critical load, the column remains straight. At the critical load, the equilibrium bifurcates, that is, the column can remain straight or can have any displacement. The critical load marks the transition from stable to unstable equilibrium.

From the design point of view, critical stress is important. It is defined as

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{n^2 \pi^2 EI}{L^2 A} \quad (1.11)$$

or

$$\sigma_{cr} = \frac{n^2 \pi^2 E}{(L/\rho)^2}, \quad (1.12)$$

where $\rho^2 = I/A$.

The mode shape of a column for different values of n is shown in Fig. 1.3.

In view of our foregoing analysis, we can make the following observations:

(i) For a column, the stress at buckling is directly proportional to the elastic modulus of the material and second moment of area, and inversely proportional to the square of the axial length. Thus, the critical load will decrease as the length increases for columns of the same material. Further, the higher the slenderness ratio, defined as L/ρ , the lower the critical load.

(ii) For a column of given cross-sectional area, the critical load can be increased by increasing I . This can be achieved by removing the material from the central portion of the column and spreading it away from the centre. Thus, for a given cross-sectional area, a hollow cross-section can resist a load higher than a solid section.

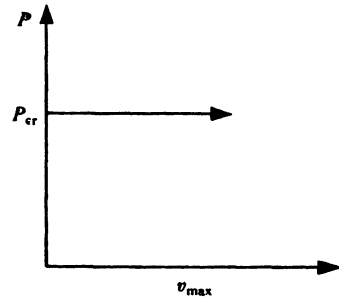


Fig. 1.2 Load-Displacement Curve.

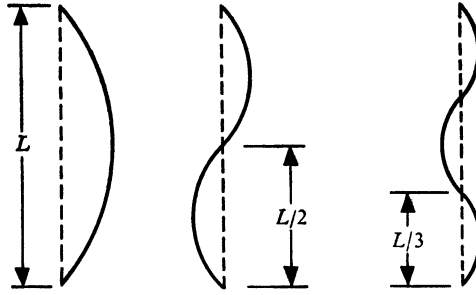


Fig. 1.3 Mode Shapes of Buckling.

(iii) For a column having specified dimensions, the critical load can be increased by choosing a material with a higher elastic modulus.

It should be noted that the equilibrium approach is characterized by the fact that there exist discrete values of the load at which additional equilibrium configurations appear in the neighbourhood of the trivial one.

1.2.2 Energy Approach

The energy approach is based on an extremum principle in mechanics which employs an energy criterion and is characterized by the condition of equilibrium in an elastic system. The energy criterion can be used to arrive at the critical load at which the response of the system ceases to be in stable equilibrium. In addition, in situations where the exact solution of the differential equation is not possible or is difficult to obtain, the energy approach can be applied to get a good approximate solution since this approach has the inherent ability to converge to the exact solution if a large number of terms are taken to represent the deflection shape.

Statement A conservative system is in equilibrium when the energy stored is equal to the work done by the external loads.

Consider a column subjected to an axial load P . When the bar bends, the two ends tend to approach each other and there is a loss of potential energy. At the same time, the column develops bending energy. The external virtual work δW_e is given by

$$\delta W_e = \sum P_i \delta r_i, \quad (1.13)$$

where δr_i represents the virtual displacement of the point of application of force P_i projected along the line of action of P_i . The strain energy δU is due to the internal work. The principle of virtual work can then be expressed as

$$\delta U - \delta W_e = 0. \quad (1.14)$$

Normally, the increment in the external work due to a virtual displacement is represented as a change in potential energy as

$$\delta V_e = -\delta W_e. \quad (1.15)$$

Accordingly, Eq. (1.14) can be rewritten as

$$\delta(U + V_e) = 0$$

or

$$\delta(\Pi_p) = 0. \quad (1.16)$$

Hence,

$$U + V_e = \Pi_p = \text{constant}. \quad (1.17)$$

The quantity $U + V_e = \Pi_p$ is referred to as the total potential of a system. Further, Eq. (1.17) is the energy criterion from which the limit of elastic stability can be determined. For the column we have just discussed, the strain energy U and the potential energy V_e are obtained as follows.

Assume that the column is slender and neglect the effect of shear deformation. Then, the strain energy stored in the column is due to bending alone, and is given by

$$U = \frac{1}{2} \int_{\text{vol}} \sigma_x \epsilon_x dv, \quad (1.18)$$

where σ_x and ϵ_x are the bending stress and bending strain along the axis of the bar. Substituting for σ_x from the constitutive equation, we have

$$U = \frac{1}{2} \int_0^L \int_A E \epsilon_x^2 dz dA. \quad (1.19)$$

The strain ϵ_x at any point in the cross-section is given by

$$\epsilon_x = y \frac{d^2 v}{dz^2}, \quad (1.20)$$

where y is the distance from the neutral axis. In view of Eq. (1.20), the strain energy U can be written as

$$U = \frac{1}{2} \int_0^L \int_A E y^2 \left(\frac{d^2 v}{dz^2} \right)^2 dA dz. \quad (1.21)$$

Integrating with respect to the area A , we get

$$U = \frac{1}{2} \int_0^L EI \left(\frac{d^2 v}{dz^2} \right)^2 dz. \quad (1.22)$$

The potential energy V_e is given by

$$V_e = -P \Delta L. \quad (1.23)$$

To evaluate V_e , it is necessary to obtain an expression for ΔL , the distance by which the ends of the column come closer as a result of bending. From Fig. 1.4, ΔL is equal to the difference between the curved length S and the straight length L , i.e.,

$$\Delta L = S - L. \quad (1.24)$$

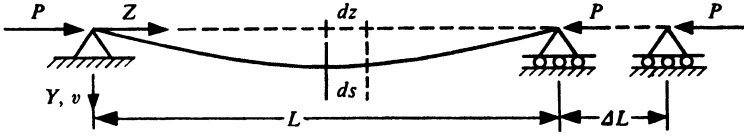


Fig. 1.4 Deformed Configuration of Column.

The curved length ds of a differential element dz is given by

$$ds^2 = dz^2 + dv^2 \quad (1.25)$$

or

$$ds = [1 + (\frac{dv}{dz})^2]^{1/2}. \quad (1.26)$$

Integrating both sides of Eq. (1.26), we obtain

$$S = \int_0^L [1 + (\frac{dv}{dz})^2]^{1/2} dz. \quad (1.27)$$

Expanding the integrand by means of the binomial theorem and retaining terms only up to an order of $(dv/dz)^2$, we get

$$\Delta L = S - L = \frac{1}{2} \int_0^L (\frac{dv}{dz})^2 dz. \quad (1.28)$$

Using Eq. (1.23), we can obtain V_e as

$$V_e = -\frac{P}{2} \int_0^L (\frac{dv}{dz})^2 dz. \quad (1.29)$$

Hence, the total potential Π_p is given by

$$\Pi_p = U + V_e = \int_0^L [\frac{EI}{2} (\frac{d^2v}{dz^2})^2 - \frac{P}{2} (\frac{dv}{dz})^2] dz. \quad (1.30)$$

For small values of P , Π_p is positive in any nontrivial admissible function $v(z)$. For sufficiently large values of P , Π_p is negative, making the equilibrium configuration unstable. Thus, the energy approach is characterized by the question: What is the value of the load for which the total potential of a perfect system ceases to be positive definite?

The critical load can be obtained from Eq. (1.30) by assuming a suitable admissible function for v and integrating it over the entire domain. For the structure considered in Section 1.2.1, $v(z)$ can be taken to be

$$v(z) = A \sin \frac{\pi z}{L}. \quad (1.31)$$

Relation (1.31) satisfies all the four boundary conditions, namely, (1.4) and (1.5). As we shall subsequently see, in most situations, a number of terms have to be taken in the

representation for v . Substituting for v and integrating it with respect to z , we get

$$P_{cr} = \pi^2 EI / L^2. \quad (1.32)$$

This result is the same as that given by Eq. (1.10) since the approximating function $v(z)$ we have taken happens to be the exact function for the differential equation (1.2). This need not be true in most situations. In fact, it is not.

1.2.3 Imperfection Approach

In our discussion so far, we have assumed that the column is perfectly straight and the loading axis passes through the centre of gravity of the section. This is not so in a real-life structure. Both imperfection of shape and small eccentricity of loading are present in an actual structure. It is therefore worthwhile that we study imperfect columns and establish the criterion for predicting the critical loads of perfect columns.

Initially bent column

Assume that the column is initially bent (see Fig. 1.5). Suppose that all the other assumptions made in Section 1.2.1 are valid. The initial deformation of the column is v_0 and its final deflected shape as measured from the straight configuration is v .

Let us assume that the initial bent shape is given by

$$v_0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{L}. \quad (1.33)$$

Since bending strains are caused by a change in curvature, the internal resistive moment at any section is

$$M = -EI \frac{d^2}{dz^2} (v - v_0). \quad (1.34)$$

Equating this to the externally applied moment Pv , we get

$$EI \frac{d^2 v}{dz^2} + Pv = EI \frac{d^2 v_0}{dz^2}. \quad (1.35)$$

Substituting for v_0 from Eq. (1.33), we obtain

$$EI \frac{d^2 v}{dz^2} + Pv = -EI \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi z}{L}. \quad (1.36)$$

The solution of Eq. (1.36) is

$$v = A \sin kz + B \cos kz + v_p, \quad (1.37)$$

where $k^2 = P/(EI)$ and v_p is the particular integral. Let

$$v_p = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi z}{L}. \quad (1.38)$$

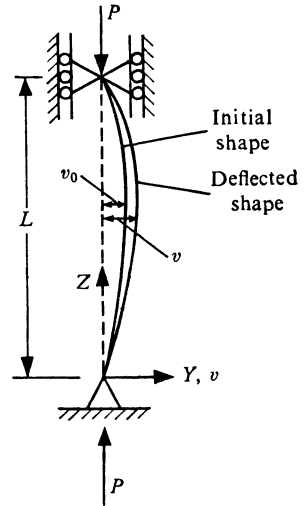


Fig. 1.5 Column with Initial Curvature.

Substituting Eq. (1.38) in Eq. (1.36), we get

$$-\sum_{n=1}^{\infty} F_n \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi z}{L} + \frac{P}{EI} \sum_{n=1}^{\infty} F_n \sin \frac{n\pi z}{L} = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi z}{L}. \quad (1.39)$$

Considering only the n -th term, we have

$$F_n = \frac{a_n}{1 - \frac{P}{EI} \frac{L^2}{n^2 \pi^2}} = \frac{a_n}{1 - \frac{P}{P_E} \frac{1}{n^2}}, \quad (1.40)$$

where $P_E = \pi^2 EI / L^2$. The total deflection v can then be written as

$$v = A \sin kz + B \cos kz + \sum \frac{a_n}{1 - \frac{P}{P_E} \frac{1}{n^2}} \sin \frac{n\pi z}{L}. \quad (1.41)$$

The arbitrary constants A and B in relation (1.41) are evaluated from the boundary conditions. The condition

$$v = 0 \quad (z = 0)$$

leads to

$$B = 0,$$

and from the condition

$$v = 0 \quad (z = L),$$

we get

$$A \sin kL = 0.$$

This implies that $A = 0$. Hence, v can be written as

$$v = \sum \frac{a_n}{1 - \frac{P}{P_E} \frac{1}{n^2}} \sin \frac{n\pi z}{L}. \quad (1.42)$$

Considering a single-term representation for the deflected shape, we can express the deflection at the midpoint of the column as

$$v(z = L/2) = \frac{a}{1 - P/P_E}. \quad (1.43)$$

As $P \rightarrow P_E$, that is, the load applied approaches the Euler load for a perfect column, the deflection at the midpoint tends to infinity. Thus, we can define the critical load of a perfect column as the load at which an imperfect column develops an infinite displacement.

At this stage, it should be noted that the problem of an initially bent column is not an eigenvalue problem since, for every load, there is a definite displacement.

Eccentrically loaded column

In many situations, the centre of gravity of the section and that of the loading may not lie along the same axis. This means that there is an eccentricity in the loading. It may exist at both ends, or at only one end, of the column. Figure 1.6 shows an eccentrically loaded column. It is assumed that the member is initially straight. The presence of moment due to eccentricity at one end gives rise to the reaction Pe/L at both ends of the column because the column is hinged, as shown. Equating the internal resistive moment and the external moment, we can write

$$EI \frac{d^2v}{dz^2} + Pv = -\frac{Pe}{L}z. \quad (1.44)$$

The boundary conditions are

$$\begin{aligned} v &= 0 & (z = 0), \\ v &= 0 & (z = L). \end{aligned} \quad (1.45)$$

Equation (1.44) is a nonhomogeneous ordinary differential equation with constant coefficients. Its solution is

$$v = A \sin kz + B \cos kz + v_p. \quad (1.46)$$

The arbitrary constants A and B here are obtained from the boundary conditions. Following a procedure similar to that described earlier in this section, we see that

$$v = \left(\frac{\sin kz}{\sin kL} - \frac{z}{L} \right) e, \quad (1.47)$$

$v \rightarrow \infty$ ($0 < z < L$) for $kL \rightarrow \pi$, i.e., when the external load corresponds to the critical load of a perfect column, the deflection tends to be very large. This observation is similar to the one made earlier in this section.

Once again, it can be noted that the problem no longer remains an eigenvalue problem.

1.2.4 Dynamic (or Vibration) Approach

We have so far been assuming that the load is gradually applied such that static equilibrium is maintained at every increment of loading. Consider now a perfect column which is disturbed laterally throughout its length. As the column starts oscillating, inertial force, in addition to static force, develops. This inertial force has therefore to be included in the equation of equilibrium. This is done by invoking D'Alembert's principle which states that a body, not in static equilibrium by virtue of some acceleration it possesses, can be brought into static equilibrium by introducing the inertia force; this force can be considered to be equivalent of the external force. This inertia force is equal to mass times the accelera-

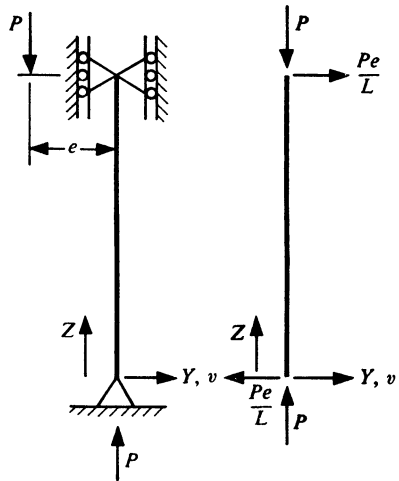


Fig. 1.6 Eccentrically Loaded Column.

tion of the body and acts through the centre of gravity of the body in the direction opposite to that of acceleration.

Differentiating Eq. (1.2) twice with respect to z and assuming that EI is constant, we get

$$EI \frac{d^4 v}{dz^4} + P \frac{d^2 v}{dz^2} = 0. \quad (1.48)$$

(In Section 1.4, we shall develop this equation from the first principles.) Equation (1.48) gives the equilibrium of static force. For a dynamic problem, following D'Alembert's principle, the equilibrium equation for the bent configuration is

$$EI \frac{\partial^4 v}{\partial z^4} + P \frac{\partial^2 v}{\partial z^2} + m_b \frac{\partial^2 v}{\partial t^2} = 0. \quad (1.49)$$

Here, we have used the partial derivatives since v is a function of z and t ; m_b defines the mass per unit length of the beam.

Consider a uniform column, hinged at both the ends. The governing equation of such a column is given by (1.49); this is a linear partial differential equation with even order derivatives, and therefore we can look for a variable separable solution in space and time. Let such a solution be given by

$$v(z, t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi z}{L}, \quad (1.50)$$

where ω_n is the natural frequency of the n -th mode. Substituting Eq. (1.50) in Eq. (1.49), we have

$$EI \frac{n^4 \pi^4}{L^4} - P \frac{n^2 \pi^2}{L^2} - m_b \omega_n^2 = 0 \quad (1.51)$$

or

$$m_b \omega_n^2 = \frac{n^2 \pi^2}{L^2} (EI \frac{n^2 \pi^2}{L^2} - P). \quad (1.52)$$

Considering only the fundamental mode, we find Eq. (1.52) reduces to

$$m_b \omega_1^2 = \frac{\pi^2}{L^2} (EI \frac{\pi^2}{L^2} - P) \quad (1.53)$$

or

$$m_b \omega_1^2 = \frac{\pi^2}{L^2} (P_{cr} - P), \quad (1.54)$$

where $P_{cr} = \pi^2 EI / L^2$ is the Euler load for the hinged end column. The three cases that now arise are (i) $\omega_1^2 = 0$, (ii) $\omega_1^2 < 0$, and (iii) $\omega_1^2 > 0$.

(i) When $P \rightarrow P_{cr}$, $\omega_1^2 \rightarrow 0$. This means that as the applied load approaches the Euler load, the natural frequency tends to 0, and is zero at $P = P_{cr}$.

(ii) If $P > P_{cr}$, $\omega_1^2 < 0$. This implies that the deflection increases with time. In such a situation, the column is said to be unstable.

(iii) If $P < P_{cr}$, $\omega_1^2 > 0$. This means that the disturbance does not die out or amplify, and the column keeps on oscillating.

Thus, as we have seen, the dynamic approach suggests a criterion for evaluating the load at which the column develops a steady configuration.

1.3 SUMMARY

We can summarize the various approaches for estimating the critical load of a column as follows:

(i) *Euler approach* Is there any configuration other than the straight configuration in which the column is in equilibrium?

(ii) *Energy approach* When is the column neutrally stable?

(iii) *Imperfection approach* When does a column with initial imperfection develop infinite displacement?

(iv) *Dynamic approach* Assuming that the column is initially set into oscillation, when does it cease to oscillate?

One of these approaches can be used to estimate the buckling load of a column, depending on the ease with which it can be applied. In what follows, we shall detail some of these techniques.

1.4 HIGHER ORDER GOVERNING EQUATION

The second order differential equation, i.e., (1.2), we derived in Section 1.2.1 cannot be applied as it is for boundary conditions other than simple supports. In this section therefore, we shall derive a general governing equation which is applicable for all types of boundary conditions.

Consider a uniform column subjected to an axial load P , as shown in Fig. 1.7a. Further, suppose that the assumptions made in Section 1.2.1 are still valid. Isolating a small element dz and indicating all the possible forces and moments (see Fig. 1.7b), we have, from the equilibrium of transverse forces,

$$V + \frac{dV}{dz} dz - V =$$

or

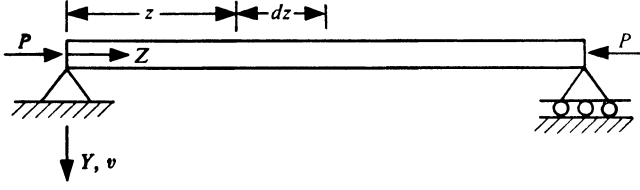
$$dV/dz = 0. \quad (1.55)$$

From moment equilibrium, we obtain

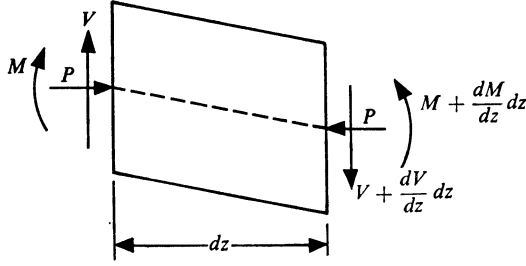
$$M - (M + \frac{dM}{dz} dz) + V dz + \frac{dV}{dz} dz + P dv = 0.$$

Simplifying, we get

$$-\frac{dM}{dz} + V + P \frac{dv}{dz} = 0. \quad (1.56)$$



(a) Column under axial load



(b) Forces acting on element

Fig. 1.7 Column with Internal Forces.

Eliminating V by differentiating Eq. (1.56) with respect to z and using Eq. (1.55), we have

$$-\frac{d^2}{dz^2}M + P\frac{d^2v}{dz^2} = 0. \quad (1.57)$$

Substituting for M from Eq. (1.1), we find that

$$\frac{d^2}{dz^2}(EI\frac{d^2v}{dz^2}) + P\frac{d^2v}{dz^2} = 0. \quad (1.58)$$

Equation (1.58) is the governing equation for any column under axial load. For a non-uniform column, it is rather difficult, if not impossible, to obtain a closed-form solution. For a uniform homogeneous column, Eq. (1.58) reduces to

$$EI\frac{d^4v}{dz^4} + P\frac{d^2v}{dz^2} = 0. \quad (1.59)$$

Unlike Eq. (1.2), which is the moment equilibrium equation, Eq. (1.59) is the transverse force equilibrium equation. The latter can be thought of as representing a beam loaded transversely, the load term q being replaced by a fictitious load given by $(-P d^2v/dz^2)$.

1.5 ANALYSIS FOR VARIOUS BOUNDARY CONDITIONS

In what follows, we shall use Eq. (1.59) to estimate the critical loads for columns with

different boundary conditions. Chajes [1] has also dealt with such problems. However, our approach here is slightly different from that he has presented.

Both ends clamped

If a column is clamped at both the ends, as shown in Fig. 1.8, the boundary conditions are

$$\begin{aligned} v &= 0 & (z = 0, L), \\ dv/dz &= 0 & (z = 0, L). \end{aligned} \quad (1.60)$$

Using a nondimensional parameter ξ , defined as $\xi = z/L$, we see that Eq. (1.59) and the boundary conditions (1.60) reduce to

$$\frac{d^4 v}{d\xi^4} + \lambda^2 \frac{d^2 v}{d\xi^2} = 0, \quad (1.61)$$

$$v = 0 = \frac{1}{L} \frac{dv}{d\xi} \quad (\xi = 0, 1), \quad (1.62)$$

where $\lambda^2 = PL^2/(EI)$ is a nondimensional buckling parameter. Since the differential equation (1.61) and the boundary conditions (1.62) are homogeneous, the problem here is an eigenvalue problem. The most general solution of Eq. (1.61) is

$$v = c_1 \cos \lambda \xi + c_2 \sin \lambda \xi + c_3 \xi + c_4. \quad (1.63)$$

The arbitrary constants c_1 , c_2 , c_3 , and c_4 here are evaluated from the boundary conditions. Applying the boundary conditions (1.62) to Eq. (1.63), the simultaneous homogeneous algebraic equations we obtain are

$$c_1 + c_4 = 0, \quad (1.64a)$$

$$c_2 \lambda + c_3 = 0, \quad (1.64b)$$

$$c_1 \cos \lambda + c_2 \sin \lambda + c_3 + c_4 = 0, \quad (1.64c)$$

$$-c_1 \lambda \sin \lambda + c_2 \lambda \cos \lambda + c_3 = 0. \quad (1.64d)$$

For a nontrivial solution or nonzero values of the constants, the determinant of the coefficients of $c_1, \dots, c_4$ given by the following matrix must vanish:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \lambda & 1 & 0 \\ \cos \lambda & \sin \lambda & 1 & 1 \\ -\lambda \sin \lambda & \lambda \cos \lambda & 1 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = 0. \quad (1.65)$$

This results in a characteristic equation, namely,

$$2(\cos \lambda - 1) + \lambda \sin \lambda = 0. \quad (1.66)$$

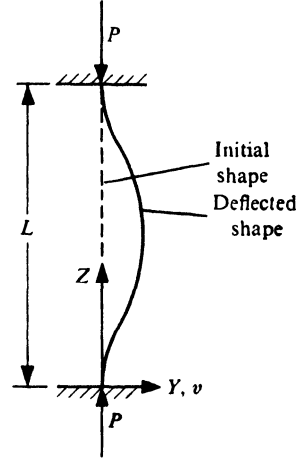


Fig. 1.8 Clamped-Clamped Column.

Equation (1.66) can be simplified, using trigonometric relations, as

$$\sin \frac{\lambda}{2} \left(\frac{\lambda}{2} \cos \frac{\lambda}{2} - \sin \frac{\lambda}{2} \right) = 0. \quad (1.67)$$

Equation (1.67) can be satisfied if (i) $\sin (\lambda/2) = 0$ or (ii) $(\lambda/2) \cos (\lambda/2) - \sin (\lambda/2) = 0$.

(i) If $\sin (\lambda/2) = 0$, then

$$\lambda/2 = n\pi \quad (n = 1, 2, 3, \dots). \quad (1.68)$$

Hence,

$$P_{cr} = \frac{4n^2\pi^2 EI}{L^2} \quad (n = 1, 2, 3, \dots). \quad (1.69)$$

Now, since $\lambda = 2n\pi$, $\sin \lambda = 0$ and $\cos \lambda = 1$. Therefore,

$$c_2 = c_3 = 0, \quad c_1 = -c_4 \quad (1.70)$$

and the deflected shape is given by

$$v = c_1(\cos 2n\pi\xi - 1). \quad (1.71)$$

The mode shapes of the column for two different values of λ are shown in Fig. 1.9. The first or the minimum critical load for $n = 1$ is four times higher than that for a simply-supported column. In other words, this column is four times stiffer than a simply-supported column.

(ii) When $\frac{\lambda}{2} \cos \frac{\lambda}{2} - \sin \frac{\lambda}{2} = 0$,

$$\frac{\lambda}{2} = \tan \frac{\lambda}{2}. \quad (1.72)$$

Equation (1.72) is a transcendental equation. Its lowest root is obtained when $\lambda/2 = 4.493$. Hence,

$$P_{cr} = 8.18\pi^2 EI/L^2. \quad (1.73)$$

The critical load given by Eq. (1.73) is higher than that obtained for case (i)—see Eq. (1.69)—and corresponds to the first antisymmetric buckling mode.

As already stated, form failure occurs when the critical load is reached. Since the critical load for the first symmetric buckling is lower than that for the antisymmetric buckling, case (i) is of practical utility.

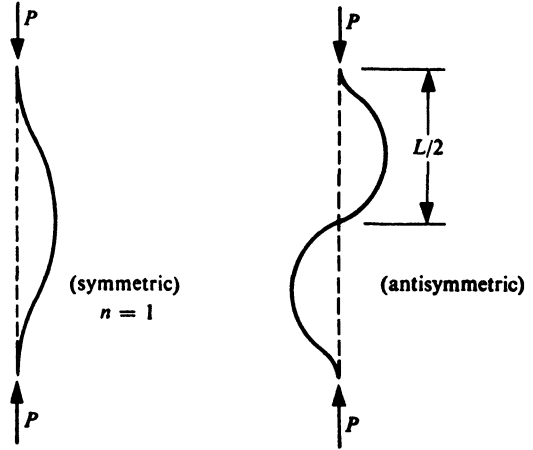


Fig. 1.9 Mode Shapes of Buckling.

One end clamped and other free

For the column shown in Fig. 1.10, the governing equation is the same as Eq. (1.61). However, the boundary conditions are

$$v = 0 = \frac{dv}{dz} \quad (z = 0), \quad (1.74)$$

$$-EI \frac{d^2v}{dz^2} = 0 \quad (z = L), \quad (1.75)$$

$$EI \frac{d^3v}{dz^3} + P \frac{dv}{dz} = 0 \quad (z = L). \quad (1.76)$$

In terms of the nondimensional parameter ξ , the boundary conditions (1.74), (1.75), and (1.76) can be written as

$$v = 0 = \frac{1}{L} \frac{dv}{d\xi} \quad (\xi = 0), \quad (1.77)$$

$$-\frac{EI}{L^2} \frac{d^2v}{d\xi^2} = 0 \quad (\xi = 1), \quad (1.78)$$

$$\frac{d^3v}{d\xi^3} + \lambda^2 \frac{dv}{d\xi} = 0 \quad (\xi = 1). \quad (1.79)$$

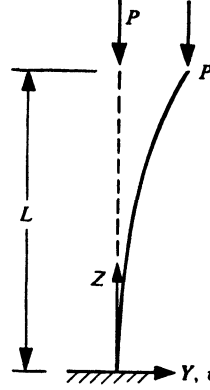


Fig. 1.10 Clamped-Free Column.

Applying these boundary conditions to Eq. (1.63), the simultaneous homogeneous algebraic equations we obtain are

$$c_1 + c_4 = 0, \quad (1.80a)$$

$$c_2\lambda + c_3 = 0, \quad (1.80b)$$

$$-c_1 \cos \lambda - c_2 \sin \lambda = 0, \quad (1.80c)$$

$$c_3 = 0. \quad (1.80d)$$

Therefore, we can conclude that $c_2 = c_3 = 0$. From Eq. (1.80c),

$$c_1 \cos \lambda = 0. \quad (1.81)$$

For a nontrivial solution, $c_1 \neq 0$. Hence,

$$\cos \lambda = 0 \Rightarrow \lambda = \frac{(2n-1)\pi}{2} \quad (n = 1, 2, 3, \dots), \quad (1.82)$$

$$P_{cr} = \frac{(2n-1)^2\pi^2 EI}{4L^2}. \quad (1.83)$$

The deflected shape is given by

$$v = c_1(\cos \lambda \xi - 1). \quad (1.84)$$

The first critical load for $n = 1$ is

$$P_{cr} = \frac{\pi^2}{4} \frac{EI}{L^2}. \quad (1.85)$$

Comparing relation (1.85) with Eq. (1.32), we observe that the critical load here is one-fourth that for a simply-supported column. In other words, a hinged end column is four times stiffer than a column with one end fixed and the other free.

One end fixed and other hinged

The boundary conditions for a column fixed at one end and hinged at the other (see Fig. 1.11) are

$$v = dv/dz = 0 \quad (z = 0), \quad (1.86)$$

$$v = -EI \frac{d^2v}{dz^2} = 0 \quad (z = L). \quad (1.87)$$

In terms of the nondimensional parameter ξ , the boundary conditions can be written as

$$v = \frac{1}{L} \frac{dv}{d\xi} = 0 \quad (\xi = 0), \quad (1.88)$$

$$v = -\frac{EI}{L^2} \frac{d^2v}{d\xi^2} = 0 \quad (\xi = 1). \quad (1.89)$$

Applying these boundary conditions to Eq. (1.63), we obtain the algebraic equations

$$c_1 + c_4 = 0, \quad (1.90a)$$

$$c_2\lambda + c_3 = 0, \quad (1.90b)$$

$$c_1 \cos \lambda + c_2 \sin \lambda + c_3\lambda + c_4 = 0, \quad (1.90c)$$

$$c_1 \cos \lambda + c_2 \sin \lambda = 0. \quad (1.90d)$$

For a nontrivial solution, the determinant of the coefficient matrix must be zero. This leads to the transcendental equation

$$c_2(\lambda - \tan \lambda) = 0,$$

and hence

$$\tan \lambda = \lambda. \quad (1.91)$$

To solve relation (1.91), we can resort to the graphical method. The smallest root we thus obtain is

$$\lambda = 4.493 \quad (1.92)$$

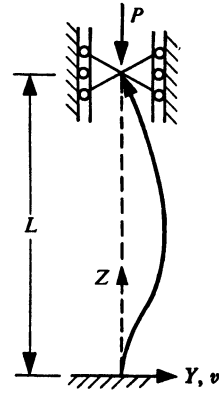


Fig. 1.11 Clamped-Hinged Column.

and the corresponding critical load is

$$P_{cr} = (20.2) \frac{EI}{L^2}. \tag{1.93}$$

Table 1.1 gives the values of the buckling parameter λ for the first three modes of buckling for different boundary conditions. Using these values, we can estimate the critical or buckling load for a uniform column.

Table 1.1 Buckling Parameter λ for Uniform Column

End condition	λ_1	λ_2	λ_3
Both ends simply-supported	π	2π	3π
One end fixed, the other end free	$\pi/2$	$3\pi/2$	$5\pi/2$
One end fixed, the other end hinged	4.493	10.903	17.221
Both ends clamped (symmetric modes only)	2π	4π	6π

Columns with elastic support

The end conditions we have considered in our foregoing discussion are ideal boundary conditions. However, in real-life structures, such ideal situations are rare. Instead, the columns are usually connected to other members, for example, as in a building frame and a steel structure, which are elastic in nature. In view of this, the member deformations interact with the column deformations. This type of support is, in general, referred to as the elastic restraint. The support reactions depend on the relative rigidities of the members.

Consider a column hinged at the lower end and elastically restrained by a beam at the upper end (see Fig. 1.12). For simplicity, assume that the length and the bending stiffness of the beam and column are the same. Assume also that there is no bending of the beam on account of the load P . Further, bending is initiated due to the column buckling. Since continuity has to be maintained at the joint B , the column rotation and the beam rotation are related to each other. The method described here is quite general and can be applied to different lengths and bending rigidities by introducing these quantities at the appropriate places.

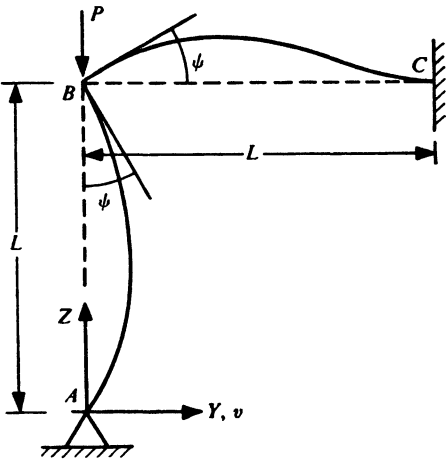


Fig. 1.12 Column with Elastic Support.

The boundary conditions for the column are

$$v = -EI \frac{d^2v}{d\xi^2} = 0 \quad (\xi = 0), \quad (1.94a)$$

$$\left. \frac{dv}{d\xi} \right|_{\xi=1} \text{ for column} = \left. \psi \right|_{\text{slope of beam}}, \quad (1.94b)$$

$$v = 0 \quad (\xi = 1). \quad (1.94c)$$

The governing equation and the general solution are given by (1.61) and Eq. (1.63), respectively. The slope of the beam at the point *B* is obtained by treating *BC* as a beam fixed at one end and hinged at the other and subjected to a bending moment. Following Peery [2], we write the bending slope as

$$\psi = ML/(4EI), \quad (1.95)$$

where *L* is the length and *EI* the bending rigidity of the beam.

For the column at $\xi = 1$,

$$M = -\frac{EI}{L^2} \frac{d^2v}{d\xi^2}. \quad (1.96)$$

Substituting for *M* in Eq. (1.95), we have

$$\psi = -\frac{1}{4} \frac{d^2v}{d\xi^2}. \quad (1.97)$$

Replacing ψ in Eq. (1.94b) and making use of the boundary conditions (1.94a) and (1.94c) along with the general solution (1.63), the equations relating the arbitrary constants we obtain are

$$c_1 + c_4 = 0, \quad (1.98a)$$

$$c_1 = 0, \quad (1.98b)$$

$$c_1 \left(\frac{\lambda^2}{4} \cos \lambda + \lambda \sin \lambda \right) + c_2 \left(\frac{\lambda^2}{4} \sin \lambda - \lambda \cos \lambda \right) - c_3 = 0, \quad (1.98c)$$

$$c_1 \cos \lambda + c_2 \sin \lambda + c_3 + c_4 = 0. \quad (1.98d)$$

After simplification, the characteristic equation for λ is

$$\tan \lambda = \frac{4\lambda}{\lambda^2 + 4}. \quad (1.99)$$

The smallest root of practical interest is $\lambda = 3.83$ and the corresponding critical load P_{cr} is

$$P_{cr} = (14.7) \frac{EI}{L^2}. \quad (1.100)$$

It is interesting to note that the value given by Eq. (1.100) lies between the values for the simply-supported and fixed-end conditions (see Table 1.1). Thus, it can be seen that

the ideal boundary conditions provide the upper and lower bounds for the buckling load of a column. From the design standpoint, a structure designed on the basis of the lower bound value is quite safe. However, in a situation where weight is a criterion, it is advisable that a realistic estimate of the load be made.

1.6 LARGE DEFLECTION OF COLUMNS

So far, we have supposed that the deflections are small. As a result, the moment-curvature relation becomes linear. Further, the magnitude of the deflection at buckling remains undetermined. To determine this magnitude, an exact expression for curvature can be used. The elastic curve thus obtained is termed as *elastica*.

Consider a simply-supported column as shown in Fig. 1.13. Except the small deflection assumption, all the other idealizations we have so far made are still valid. The column is straight and the material is assumed to be homogeneous and elastic. Taking the coordinate axes as shown in Fig. 1.13 and measuring s along the column from the point A , we find the exact expression for the curvature is $d\theta/ds$. As we know, the internal resistive moment of a column is the product of bending rigidity and curvature. Equating this resistive moment with the external moment, we get

$$EI \frac{d\theta}{ds} + Pv = 0. \quad (1.101)$$

Equation (1.101) is a nonlinear differential equation and for such an equation, it is rather difficult, if not impossible, to obtain a general solution. The analysis presented here follows that given by Timoshenko and Gere [3].

Differentiating Eq. (1.101) with respect to s and replacing dv/ds by $\sin \theta$, we get

$$\frac{d^2\theta}{ds^2} + \frac{P}{EI} \sin \theta = 0. \quad (1.102)$$

In terms of the nondimensional parameter η , defined as $s = L\eta$, Eq. (1.102) can be written as

$$\frac{d^2\theta}{d\eta^2} + k^2 \sin \theta = 0, \quad (1.103)$$

where $k^2 = PL^2/(EI)$. Multiplying both sides of Eq. (1.103) by $d\theta$ and integrating, we have

$$\int \frac{d^2\theta}{d\eta^2} d\theta + \int k^2 \sin \theta d\theta = 0. \quad (1.104)$$

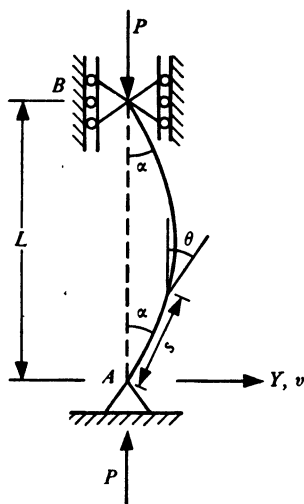


Fig. 1.13 Large Deflection of Column (after Chajes [1]).

This equation can be expressed in the form

$$\frac{1}{2} \int \frac{d}{d\eta} \left(\frac{d\theta}{d\eta} \right)^2 d\eta + k^2 \int \sin \theta d\theta = 0. \quad (1.105)$$

Carrying out the integration, we get

$$\frac{1}{2} \left(\frac{d\theta}{d\eta} \right)^2 - k^2 \cos \theta = c. \quad (1.106)$$

The constant of integration c is obtained from the condition

$$c = -k^2 \cos \alpha \quad (1.107)$$

and Eq. (1.106) then becomes

$$\left(\frac{d\theta}{d\eta} \right)^2 = 2k^2 (\cos \theta - \cos \alpha)$$

or

$$\frac{d\theta}{d\eta} = \pm \sqrt{2} k (\cos \theta - \cos \alpha)^{1/2}. \quad (1.108)$$

Only the negative sign is retained in Eq. (1.108) since the curvature increases as θ decreases. Therefore,

$$d\eta = - \frac{d\theta}{\sqrt{2} k (\cos \theta - \cos \alpha)^{1/2}}. \quad (1.109)$$

Integrating both sides, we have

$$1 = \int_0^1 d\eta = - \int_{\alpha}^{-\theta} \frac{d\theta}{\sqrt{2} k (\cos \theta - \cos \alpha)^{1/2}}$$

or

$$1 = \int_{-\alpha}^{\theta} \frac{d\theta}{\sqrt{2} k (\cos \theta - \cos \alpha)^{1/2}}. \quad (1.110)$$

Equation (1.110), when simplified, gives

$$1 = \frac{1}{2k} \int_{-\alpha}^{\theta} \frac{d\theta}{[\sin^2(\alpha/2) - \sin^2(\theta/2)]^{1/2}}. \quad (1.111)$$

This integral can be simplified by using the notation $p = \sin(\alpha/2)$, introducing a new variable ϕ , defined by

$$\sin \frac{\theta}{2} = p \sin \phi = \sin \frac{\alpha}{2} \sin \phi, \quad (1.112)$$

and replacing the θ -terms by the corresponding ϕ -terms. From Eq. (1.112),

$$d\theta = \frac{2 \sin(\alpha/2) \cos \phi}{\cos(\theta/2)} d\phi = \frac{2p \cos \phi}{\sqrt{1 - p^2 \sin^2 \phi}} d\phi. \quad (1.113)$$

Further, as θ varies from $-\alpha$ to α , $\sin \phi$ varies from -1 to 1 , i.e., ϕ varies from $-\pi/2$ to $\pi/2$. Equation (1.111) can be written in the form

$$1 = \frac{2}{k} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - p^2 \sin^2 \phi}} \quad (1.114)$$

or

$$1 = \frac{2K(p)}{k}. \quad (1.115)$$

The integral in Eq. (1.114) is known as a *complete elliptic integral of the first kind* and is designated as $K(p)$. Substituting for k , we can write Eq. (1.115) as

$$L = \frac{2K(p)}{\sqrt{P/EI}}. \quad (1.116)$$

For a given value of $p = \sin(\alpha/2)$, $K(p)$ can be obtained from the table of elliptic integrals.

If the deflection of the bar is small, α and p will also be small and the magnitude of the term $p^2 \sin^2 \phi$ in Eq. (1.114) will be negligible as compared with unity. We then have

$$1 = \frac{2}{k} \int_0^{\pi/2} d\phi = \frac{1}{k} \pi, \quad (1.117)$$

$$P = P_{cr} = \pi^2 EI / L^2. \quad (1.118)$$

Thus, for a small deflection, the linear theory and the nonlinear theory lead to the same result.

As the value of α increases, the integral $K(p)$ and the load P also increase. As an example, take $\alpha = 60^\circ$ and $p = \sin(\alpha/2) = 1/2$. From the table of elliptic integrals, we have, for $p = 1/2$, $K(p) = 1.686$. Substituting this value for $K(p)$ in Eq. (1.116), we get

$$L = \frac{2 \times 1.686}{\sqrt{P/EI}} \quad (1.119)$$

or

$$P = 11.37 EI / L^2. \quad (1.120)$$

Taking the ratio of P to P_{cr} , we have

$$P/P_{cr} = 11.37/\pi^2 = 1.152. \quad (1.121)$$

Thus, a load 15.2 per cent more than the Euler load will produce a deflection corresponding to an angular deflection of 60° at the ends of the column measured with respect to the vertical. This means that if a provision is made for a large deflection, a column can be made to withstand a large load.

Equation (1.12) gives the stress at buckling, based on the linear theory. Allowing for a large deformation implies that a column is permitted to bend. Hence, as the bending increases, at any cross-section of the column, the bending and axial stresses act. As a result, these stresses may cross the elastic limit of the column material long before the deformation becomes large.

1.7 BUCKLING OF LAMINATED COMPOSITE COLUMNS

Metallic materials are isotropic at the macroscopic level. In an isotropic material, it is impossible to achieve an enhanced strength or modulus in any direction. A composite material (e.g., glass fibre in epoxy matrix and boron fibre in epoxy matrix), on the other hand, has high strength and high modulus. Since, invariably, an optimal structure may not have the same modulus throughout its length, composite materials are increasingly being used in structural design because their modulus and strength can be varied as required. Also, the weight saving in a composite material structure over a conventional metallic structure can be substantial. What is more, unlike an isotropic material, the plies of a laminate can be oriented such that the laminate meets both the load and direction requirements.

1.7.1 Governing Equation

Figure 1.14 shows a laminated column along with the coordinate system. In deriving the governing equation for an isotropic column, we assumed that the middle plane does not stretch and that there is no coupling between the bending and inplane deformations. However, in an anisotropic column (column of composite material), coupling exists. Following the small deflection theory, we can write the strain-displacement relations for a uniaxial member as

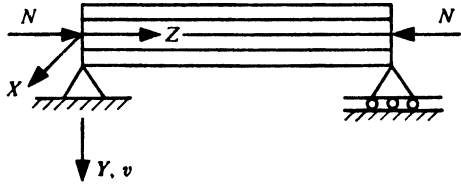


Fig. 1.14 Laminated Composite Column.

$$\epsilon_z = \frac{\partial \omega_0}{\partial z} - y \frac{\partial^2 v}{\partial z^2}, \quad \epsilon_x = 0, \quad \gamma_{xz} = 0, \quad (1.122a)$$

where ω_0 is the inplane deflection and y is the distance of the surface measured from any reference point. The constitutive equations for the k -th lamina can be written, following Eq. (I.20)—see Appendix I—, as

$$\begin{aligned} \sigma_z^k &= Q_{11}^k \epsilon_z, \\ \sigma_x^k &= Q_{12}^k \epsilon_z = \frac{Q_{12}^k}{Q_{11}^k} \sigma_z^k, \\ \tau_{xy}^k &= Q_{16}^k \epsilon_z = \frac{Q_{16}^k}{Q_{11}^k} \sigma_z^k. \end{aligned} \quad (1.122b)$$

The contribution of this lamina to the stress resultants is given by

$$\begin{aligned} N_z^k &= \int_{h_{k-1}}^{h_k} \sigma_z^k dx, & N_x^k &= \int_{h_{k-1}}^{h_k} \sigma_x^k dx, & N_{xy}^k &= \int_{h_{k-1}}^{h_k} \tau_{xy}^k dx, \\ M_z^k &= \int_{h_{k-1}}^{h_k} \sigma_z^k x dx, & M_x^k &= \int_{h_{k-1}}^{h_k} \sigma_x^k x dx, & M_{xy}^k &= \int_{h_{k-1}}^{h_k} \tau_{xy}^k x dx. \end{aligned} \quad (1.122c)$$

Substituting ϵ_x from Eqs. (1.122a) in Eqs. (1.122b) and then substituting the result in Eqs. (1.122c) and integrating, we get

$$\begin{aligned} N_z &= A_{11} \frac{\partial \omega_0}{\partial z} - B_{11} \frac{\partial^2 v}{\partial z^2}, \\ N_x &= A_{12} \frac{\partial \omega_0}{\partial z} - B_{12} \frac{\partial^2 v}{\partial z^2}, \end{aligned} \quad (1.123)$$

$$\begin{aligned} N_{zx} &= A_{16} \frac{\partial \omega_0}{\partial z} - B_{16} \frac{\partial^2 v}{\partial z^2}, \\ M_z &= B_{11} \frac{\partial \omega_0}{\partial z} - D_{11} \frac{\partial^2 v}{\partial z^2}, \\ M_x &= B_{12} \frac{\partial \omega_0}{\partial z} - D_{12} \frac{\partial^2 v}{\partial z^2}, \end{aligned} \quad (1.124)$$

$$M_{zx} = B_{16} \frac{\partial \omega_0}{\partial z} - D_{16} \frac{\partial^2 v}{\partial z^2}.$$

For a stability problem, $N_z = N$ and $M_x = 0$. The first of Eqs. (1.124) yields

$$\frac{\partial \omega_0}{\partial z} = \frac{D_{11}}{B_{11}} \frac{\partial^2 v}{\partial z^2}. \quad (1.125)$$

Substituting this equation in the first of Eqs. (1.123), we find that

$$\frac{\partial^2 v}{\partial z^2} = \frac{B_{11}}{A_{11}D_{11} - B_{11}^2} N. \quad (1.126)$$

For the surface to remain flat, $\partial^2 v / \partial z^2$ must be zero. This implies that the quantity $B_{11} / (A_{11}D_{11} - B_{11}^2)$ must be zero for all values of N . Assume that the column buckles elastically and the moment $M_x(z) = Nv(z)$. Further, since the curvature is negative, the governing equation can be obtained by putting $N_z = 0$ in the first of Eqs. (1.123) and then substituting in the result so obtained the expression for $\partial \omega_0 / \partial z$ [given by the first of Eqs. (1.124)] as

$$\frac{d^2 v}{dz^2} + \frac{A_{11}}{B_{11}^2 - A_{11}D_{11}} Nv = 0. \quad (1.127)$$

Equation (1.127) is valid for a laminated anisotropic Euler column. In terms of the non-dimensional parameter ξ , defined as $\xi = z/L$, Eq. (1.127) can be written as

$$\frac{d^2 v}{d\xi^2} + \frac{A_{11}}{B_{11}^2 - A_{11}D_{11}} L^2 Nv = 0. \quad (1.128)$$

Equation (1.128) is of the same form as Eq. (1.2), and hence its solution can be written as

$$v = A \sin \lambda \xi + B \cos \lambda \xi, \quad (1.129)$$

where

$$\lambda^2 = \frac{A_{11}L^2}{B_{11}^2 - A_{11}D_{11}}$$

and A , B are arbitrary constants to be evaluated from the boundary conditions. If the column is assumed to be hinged at both the ends, the boundary conditions are given by (1.5). Following a procedure similar to that in Section 1.2.1, the critical load N for the first mode of buckling we obtain is

$$N_{cr} = \frac{B_{11}^2 - A_{11}D_{11}}{A_{11}} \frac{\pi^2}{L^2}. \quad (1.130)$$

It should be noted that if $(B_{11}^2 - A_{11}D_{11})/A_{11}$ is replaced by the isotropic flexural rigidity EI , Eq. (1.130) would then correspond to the critical load for the Euler column. Thus, we can conclude that a good estimate of the critical load for a composite column with boundary conditions other than those considered in this section can be obtained by replacing EI in the expression for the critical load for an isotropic column by the corresponding term.

1.8 APPROXIMATE TECHNIQUES

The differential equation approach can be used mostly for uniform columns with simple boundary conditions. This technique fails when the geometry of a column cross-section varies with column length or when the boundary conditions are complicated. In such a situation, an attempt can be made to apply one of the approximate techniques, e.g., the energy approach (see Section 1.2.2), which have the inherent ability to converge to the exact solution. Further, the energy approach, when used in conjunction with the calculus of variations, leads to the exact differential equation and the associated boundary conditions (see Section 1.8.3).

In what follows, we shall discuss the approximate methods based on total potential.

1.8.1 Timoshenko's Method

According to Timoshenko and Gere [3], straight configuration is stable if Π_p [defined by Eq. (1.30)] > 0 and unstable if $\Pi_p < 0$. A critical condition is reached when $\Pi_p = 0$. This leads to

$$\frac{1}{2} \int_0^L EI \left(\frac{d^2v}{dz^2} \right)^2 dz = \frac{P_{cr}}{2} \int_0^L \left(\frac{dv}{dz} \right)^2 dz, \quad (1.131)$$

where $P = P_{cr}$ at buckling. Therefore, to obtain P_{cr} , we need to assume a suitable displacement function v . Such a function is so chosen that it satisfies both geometric and natural boundary conditions. However, it should be noted that satisfying the boundary conditions does not necessarily mean that an exact solution can be obtained.

EXAMPLE 1.1 (Critical load of cantilever column) Consider a uniform column under a constant load P (Fig. 1.15). As the load is increased, the work done by the force P gets stored in the structure as bending energy. The boundary conditions here are

$$\begin{aligned} v &= 0 & (z = 0), \\ dv/dz &= 0 & (z = 0), \end{aligned} \quad (1.132)$$

$$EI \frac{d^2 v}{dz^2} = 0 \quad (z = L), \quad (1.133)$$

$$EI \frac{d^3 v}{dz^3} + P \frac{dv}{dz} = 0 \quad (z = L).$$

Equations (1.132) are referred to as the geometric or kinematic boundary conditions and Eqs. (1.133) are known as the natural or dynamic boundary conditions.

Let us now choose a function for $v(z)$, namely,

$$v(z) = Az^2. \quad (1.134)$$

As can be seen, this satisfies only the geometric boundary conditions, i.e., (1.132). Substituting this in Eq. (1.131) and integrating, we get

$$P_{cr} = 3EI/L^2 \quad (1.135)$$

which is 21.3 per cent higher than the exact value [given by Eq. (1.85)].

Consider another shape function, namely,

$$v(z) = Az^2(3L - z). \quad (1.136)$$

This satisfies the boundary conditions (1.132) and the first of the boundary conditions (1.133). Now,

$$\begin{aligned} d^2 v/dz^2 &= -6Az + 6AL, \\ dv/dz &= -3Az^2 + 6ALz. \end{aligned} \quad (1.137)$$

Substituting for $d^2 v/dz^2$ and dv/dz in Eq. (1.131), we get

$$\frac{EI}{2} \int_0^L (36A^2 z^2 + 36A^2 L^2 - 72zL) dz = \frac{P_{cr}}{2} \int_0^L (9A^2 z^4 + 36A^2 L^2 z^2 - 36A^2 L z^3) dz. \quad (1.138)$$

Equation (1.138) on integration and simplification results in

$$P_{cr} = (2.5) \frac{EI}{L^2}. \quad (1.139)$$

This value is about 1.3 per cent higher than the exact value [given by Eq. (1.85)].

Consider yet another shape function, that is,

$$v(z) = A(1 - \cos \frac{\pi z}{2L}). \quad (1.140)$$

This satisfies the geometric as well as natural boundary conditions. Following the same procedure as just described, we obtain

$$P_{cr} = \pi^2 EI / (4L^2). \quad (1.141)$$

This, of course, is the exact solution, because the shape function we have chosen happens to be the exact shape function.

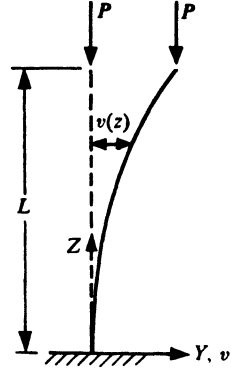


Fig. 1.15 Uniform Cantilever Column.

It should be noted that the energy approach gives a value higher than the exact. However, by choosing a proper shape function when using this approach, we can minimize the difference.

EXAMPLE 1.2 (Critical load of simply-supported column) Consider the simply-supported column shown in Fig. 1.1. The boundary conditions are (1.4) and (1.5).

Assume that $v(z)$ is given by

$$v(z) = A(L - z)z. \quad (1.142)$$

This shape function satisfies the condition that the displacements are zero at $z = 0, L$. Then, from Eq. (1.131), we have

$$P_{cr} = 12EI/L^2. \quad (1.143)$$

This value is higher than the exact value [given by Eq. (1.32)] by 21.3 per cent.

If we now choose a function that satisfies the natural boundary conditions as well, we can expect a better solution. Let

$$v(z) = A(z^4 - 2Lz^3 + L^3z) \quad (1.144)$$

be the shape function. Then,

$$\begin{aligned} dv/dz &= A(4z^3 - 6Lz^2 + L^3), \\ d^2v/dz^2 &= A(12z^2 - 12Lz). \end{aligned} \quad (1.145)$$

Substituting Eqs. (1.145) in Eq. (1.131) and integrating, we have

$$P_{cr} = (9.88) \frac{EI}{L^2} \quad (1.146)$$

which is approximately 0.13 per cent higher than the exact load [given by Eq. (1.32)].

EXAMPLE 1.3 (Buckling of column with stepped cross-section) Consider a stepped column as shown in Fig. 1.16. We can analyze this structure by using the differential equation approach, but this approach is quite involved. Since the energy technique leads to a good approximate solution, we shall apply it instead.

As already stated, a function of the type (1.140) satisfies both kinematic and natural boundary conditions. The strain energy U due to bending and the potential energy V_e are given by

$$U = \frac{1}{2} \int_0^L EI \left(\frac{d^2v}{dz^2} \right)^2 dz, \quad (1.22)$$

$$V_e = -\frac{P}{2} \int_0^L \left(\frac{dv}{dz} \right)^2 dz. \quad (1.29)$$

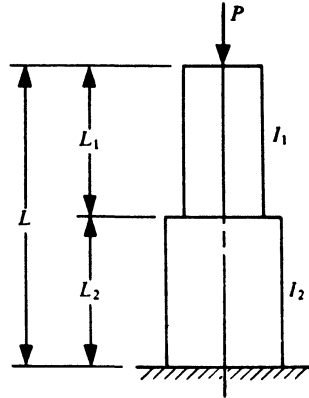


Fig. 1.16 Stepped Cantilever Column.

Now, the strain energy U for the given column is

$$\begin{aligned}
 U &= \int_0^{L_1} \frac{EI_2}{2} A^2 \left(\frac{\pi}{2L} \right)^4 \cos^2 \frac{\pi z}{2L} dz + \int_{L_1}^L \frac{EI_1}{2} A^2 \left(\frac{\pi}{2L} \right)^4 \cos^2 \frac{\pi z}{2L} dz \\
 &= \frac{EI_2}{2} A^2 \left(\frac{\pi}{2L} \right)^4 \frac{1}{2} \int_0^{L_1} (1 + \cos \frac{2\pi z}{2L}) dz + \frac{EI_1}{2} A^2 \left(\frac{\pi}{2L} \right)^4 \frac{1}{2} \int_{L_1}^L (1 + \cos \frac{2\pi z}{2L}) dz \\
 &= \frac{EI_2}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [z + \sin \frac{2\pi z}{2L} \frac{2L}{2\pi}]_0^{L_1} + \frac{EI_1}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [z + \sin \frac{2\pi z}{2L} \frac{2L}{2\pi}]_{L_1}^L \\
 &= \frac{EI_2}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [L_2 + \frac{L}{\pi} \sin \frac{\pi L_2}{L}] + \frac{EI_1}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [(L - L_2) - \frac{L}{\pi} \sin \frac{\pi L_2}{L}] \\
 &= \frac{EI_2}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [L_2 + \frac{L}{\pi} \sin \frac{\pi L_2}{L}] + \frac{EI_1}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [L_1 - \frac{L}{\pi} \sin \frac{\pi L_2}{L}] \\
 &= \frac{EI_2}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [L_2 + \frac{I_1}{I_2} L_1 + \frac{L}{\pi} \sin \frac{\pi L_2}{L} (1 - \frac{I_1}{I_2})]. \tag{1.147}
 \end{aligned}$$

Further, the potential energy V_e for the given column is

$$\begin{aligned}
 V_e &= -\frac{P}{2} \int_0^L A^2 \left(\frac{\pi}{2L} \right)^2 \sin^2 \frac{\pi z}{2L} dz \\
 &= -\frac{P}{4} A^2 \left(\frac{\pi}{2L} \right)^2 \int_0^L (1 - \cos \frac{2\pi z}{2L}) dz \\
 &= -\frac{P}{4} A^2 \left(\frac{\pi}{2L} \right)^2 [z - \sin \frac{2\pi z}{2L} \frac{2L}{2\pi}]_0^L \\
 &= -\frac{P}{4} A^2 \left(\frac{\pi}{2L} \right)^2 L. \tag{1.148}
 \end{aligned}$$

At buckling,

$$U = -V_e. \tag{1.149}$$

Thus,

$$\frac{EI_2}{4} A^2 \left(\frac{\pi}{2L} \right)^4 [L_2 + \frac{I_1}{I_2} L_1 + \frac{L}{\pi} (1 - \frac{I_1}{I_2}) \sin \frac{\pi L_2}{L}] = \frac{P_{cr}}{4} A^2 \left(\frac{\pi}{2L} \right)^2 L. \tag{1.150}$$

Simplifying, we get

$$P_{cr} = \frac{\pi^2 EI_2}{4L^2} \frac{1}{[\frac{L_2}{L} + \frac{L_1}{L} \frac{I_1}{I_2} + \frac{1}{\pi} (1 - \frac{I_1}{I_2}) \sin \frac{\pi L_2}{L}]^{-1}}. \tag{1.151}$$

We can also obtain the critical load by replacing the curvature term in Eq. (1.22) by the moment term using Eq. (1.1). Thus,

$$U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dz. \tag{1.152}$$

For the given column,

$$M = -P(A - v). \quad (1.153)$$

Substituting for v in Eq. (1.153), we can write M as

$$M = -AP \cos \frac{\pi z}{2L}. \quad (1.154)$$

Therefore,

$$U = \frac{1}{2} \int_0^L \frac{1}{EI_2} A^2 P^2 \cos^2 \frac{\pi z}{2L} dz + \frac{1}{2} \int_0^L \frac{1}{EI_1} A^2 P^2 \cos^2 \frac{\pi z}{2L} dz. \quad (1.155)$$

The expression for V_c remains unchanged and is given by Eq. (1.148). Simplifying Eq. (1.155), we get

$$U = \frac{P^2 A^2}{4EI_2} [L_2 + \frac{I_2}{I_1} L_1 + \frac{L}{2\pi} (1 - \frac{I_2}{I_1}) \sin \frac{\pi L_2}{L}]. \quad (1.156)$$

Hence,

$$P_{cr} = \frac{\pi^2 EI_2}{4L^2} \frac{1}{[\frac{L_2}{L} + \frac{I_2}{I_1} \frac{L_1}{L} + \frac{1}{2\pi} (1 - \frac{I_2}{I_1}) \sin \frac{\pi L_2}{L}]}. \quad (1.157)$$

As we can see, the expressions within the square brackets in Eqs. (1.151) and (1.157) are not identical. In view of this, the critical loads we have obtained using the two alternatives are not the same. Normally, the lower of the two is taken as the critical load for purposes of design. Of these two loads, which one would be closer to the exact value would depend on how closely the curve we choose represents the deflection or the moment. However, it can be seen that, for a uniform column, both the alternatives would yield the same result.

1.8.2 Rayleigh-Ritz Method

According to the neutral equilibrium concept, the critical load is the load at which a system can be in equilibrium even when it is in a slightly bent configuration. Thus, the problem of finding the critical load for a system is equivalent to that of finding the deflected shape for which the system is in equilibrium. This can be done by making the first variation of total potential to vanish. This is expressed by Eq. (1.16). We shall illustrate the Rayleigh-Ritz method by again considering Example 1.1.

EXAMPLE 1.4 (Critical load of cantilever column) Let us choose a function for $v(z)$ as

$$v(z) = Az^2. \quad (1.158)$$

This satisfies the geometric boundary conditions (1.132) for the cantilever column (Example 1.1).

The strain energy due to the bending is

$$U = \frac{EI}{2} \int_0^L \left(\frac{d^2 v}{dz^2} \right)^2 dz = \frac{EI}{2} \int_0^L 4A^2 dz = 2EIA^2 L. \quad (1.159)$$

The potential energy due to the external load is

$$V_e = -\frac{P}{2} \int_0^L \left(\frac{dv}{dz}\right)^2 dz = -\frac{P}{2} \int_0^L 4A^2 z^2 dz = -\frac{2}{3} P A^2 L^3. \quad (1.160)$$

The total potential Π_p is given by

$$\Pi_p = U + V_e = 2EI A^2 L - \frac{2}{3} P A^2 L^3. \quad (1.161)$$

The total potential is a function of the single parameter A . Thus, the expression

$$\delta(U + V_e) = 0$$

can be replaced by

$$\frac{d}{dA}(U + V_e)\delta A = 0. \quad (1.162)$$

Since δA is arbitrary and cannot be zero for a nontrivial solution,

$$\frac{d}{dA}(U + V_e) = 0. \quad (1.163)$$

Substituting for U and V_e in Eq. (1.163) and carrying out the differentiation, we obtain

$$A(EI - \frac{2}{3}PL^2) = 0. \quad (1.164)$$

Since $A \neq 0$,

$$P_{cr} = 3EI/L^2. \quad (1.165)$$

This result is the same as given by Eq. (1.135) and is approximately 21.3 per cent higher than the exact value [given by Eq. (1.85)].

We can obtain the exact buckling load by assuming a displacement function with infinite terms. However, we can get a good approximate solution if we replace an infinite-degree-of-freedom system by a finite-degree-of-freedom system. For our example, we can achieve a solution better than that given by Eq. (1.165) if we consider a two-term displacement function, namely,

$$v = Az^2 + Bz^3.$$

The first and the second derivatives of v are

$$dv/dz = 2Az + 3Bz^2,$$

$$d^2v/dz^2 = 2A + 6Bz.$$

The strain energy due to the bending is

$$\begin{aligned} U &= \frac{EI}{2} \int_0^L \left(\frac{d^2v}{dz^2}\right)^2 dz = \frac{EI}{2} \int_0^L (4A^2 + 36B^2z^2 + 24ABz) dz \\ &= 2EIL(A^2 + 3B^2L^2 + 3ABL) \end{aligned} \quad (1.166)$$

and the potential energy due to the external load is

$$\begin{aligned} V_e &= -\frac{P}{2} \int_0^L \left(\frac{dv}{dz}\right)^2 dz = -\frac{P}{2} \int_0^L (4A^2z^2 + 12ABz^3 + 9B^2z^4) dz \\ &= -\frac{PL^3}{2} \left(\frac{4}{3}A^2 + 3ABL + \frac{9}{5}B^2L^2\right). \end{aligned} \quad (1.167)$$

Adding Eqs. (1.166) and (1.167), we get

$$\Pi_p = U + V_e = 2EIL(A^2 + 3B^2L^2 + 3ABL) - \frac{PL^3}{2} \left(\frac{4}{3}A^2 + 3ABL + \frac{9}{5}B^2L^2\right). \quad (1.168)$$

The total potential Π_p is a function of the two variables A and B . Hence,

$$\delta(U + V_e) \equiv \frac{\partial}{\partial A}(U + V_e) \delta A + \frac{\partial}{\partial B}(U + V_e) \delta B = 0. \quad (1.169)$$

Since δA and δB are arbitrary, Eq. (1.169) can be satisfied only if

$$\frac{\partial}{\partial A}(U + V_e) = 0, \quad \frac{\partial}{\partial B}(U + V_e) = 0. \quad (1.170)$$

Differentiating Eq. (1.168) with respect to A, B , we obtain

$$\begin{aligned} 2EIL(2A + 3BL) - \frac{PL^3}{2} \left(\frac{8}{3}A + 3BL\right) &= 0, \\ 2EIL(3AL + 6BL^2) - \frac{PL^3}{2} (3AL + \frac{18}{5}BL^2) &= 0. \end{aligned} \quad (1.171)$$

Introducing the notation $\beta = PL^2/(EI)$, we can rewrite Eqs. (1.171) as

$$\begin{aligned} (24 - 8\beta)A + (36L - 9\beta L)B &= 0, \\ (20 - 5\beta)A + (40L - 6\beta L)B &= 0. \end{aligned} \quad (1.172)$$

Equations (1.172) are a set of simultaneous homogeneous algebraic equations in A and B . Hence, for a nontrivial solution, the determinant of the coefficient matrix of A and B must be zero. Thus,

$$\begin{vmatrix} 24 - 8\beta & 36L - 9\beta L \\ 20 - 5\beta & 40L - 6\beta L \end{vmatrix} = 0. \quad (1.173)$$

The characteristic equation for β is

$$3\beta^2 - 104\beta + 240 = 0. \quad (1.174)$$

The roots of this equation are

$$\beta_1 = 2.49, \quad \beta_2 = 32.19. \quad (1.175)$$

Of these, the smaller is $\beta = 2.49$. From this, we obtain

$$P_{cr} = 2.49EI/L^2. \quad (1.176)$$

This solution differs from the exact value [given by (1.85)] by 1 per cent. Thus, we can improve the solution by increasing the number of terms in the deflection function. The second root of Eq. (1.174) corresponds to a higher mode. But a two-term representation for such a mode would give a crude approximation. However, in this situation too, we can improve the accuracy by considering more terms in the representation for displacement.

EXAMPLE 1.5 (Buckling of column with stepped cross-section) Consider Example 1.3 with a two-term representation for displacement as

$$v = A(1 - \cos \frac{\pi z}{2L}) + B(1 - \cos \frac{3\pi z}{2L}). \quad (1.177)$$

With this representation,

$$\frac{dv}{dz} = A(\frac{\pi}{2L}) \sin \frac{\pi z}{2L} + B(\frac{3\pi}{2L}) \sin \frac{3\pi z}{2L}, \quad (1.178)$$

$$\frac{d^2v}{dz^2} = A(\frac{\pi}{2L})^2 \cos \frac{\pi z}{2L} + B(\frac{3\pi}{2L})^2 \cos \frac{3\pi z}{2L}. \quad (1.179)$$

Substituting for d^2v/dz^2 in Eq. (1.22), we get

$$\begin{aligned} U &= \int_0^{L_1} \frac{EI_2}{2} (\frac{\pi}{2L})^4 (A^2 \cos^2 \frac{\pi z}{2L} + 81B^2 \cos^2 \frac{3\pi z}{2L} + 18AB \cos \frac{\pi z}{2L} \cos \frac{3\pi z}{2L}) dz \\ &\quad + \int_{L_1}^L \frac{EI_1}{2} (\frac{\pi}{2L})^4 (A^2 \cos^2 \frac{\pi z}{2L} + 81B^2 \cos^2 \frac{3\pi z}{2L} + 18AB \cos \frac{\pi z}{2L} \cos \frac{3\pi z}{2L}) dz \\ &= \int_0^{L_1} \frac{EI_2}{4} (\frac{\pi}{2L})^4 [A^2(1 + \cos \frac{2\pi z}{2L}) + 81B^2(1 + \cos \frac{6\pi z}{2L}) \\ &\quad + 18AB(\cos \frac{4\pi z}{2L} + \cos \frac{2\pi z}{2L})] dz + \int_{L_1}^L \frac{EI_1}{4} (\frac{\pi}{2L})^4 [A^2(1 + \cos \frac{2\pi z}{2L}) \\ &\quad + 81B^2(1 + \cos \frac{6\pi z}{2L}) + 18AB(\cos \frac{4\pi z}{2L} + \cos \frac{2\pi z}{2L})] dz \\ &= \frac{EI_2}{4} (\frac{\pi}{2L})^4 [A^2(z + \sin \frac{\pi z}{L} \frac{L}{\pi}) + 81B^2(z + \sin \frac{3\pi z}{2L} \frac{L}{3\pi}) \\ &\quad + 18AB(\sin \frac{2\pi z}{L} \frac{L}{2} + \sin \frac{\pi z}{L} \frac{L}{\pi})]_0^{L_1} + \frac{EI_1}{4} (\frac{\pi}{2L})^4 [A^2(z + \sin \frac{\pi z}{L} \frac{L}{\pi}) \\ &\quad + 81B^2(z + \sin \frac{3\pi z}{2L} \frac{L}{3\pi}) + 18AB(\sin \frac{2\pi z}{L} \frac{L}{2\pi} + \sin \frac{\pi z}{L} \frac{L}{\pi})]_{L_1}^L \\ &= \frac{EI_2}{4} (\frac{\pi}{2L})^4 [A^2(L_2 + \sin \frac{\pi L_2}{L} \frac{L}{\pi}) + 81B^2(L_2 + \sin \frac{3\pi L_2}{2L} \frac{L}{3\pi}) \\ &\quad + 18AB(\frac{L}{\pi} (\frac{1}{2} \sin \frac{2\pi L_2}{L} + \sin \frac{\pi L_2}{L}))] + \frac{EI_1}{4} (\frac{\pi}{2L})^4 [A^2(L_1 - \frac{L}{\pi} \sin \frac{\pi L_2}{L}) \\ &\quad + 81B^2(L_1 - \frac{L}{3\pi} \sin \frac{3\pi L_2}{L}) + 18AB \frac{L}{\pi} (-\frac{1}{2} \sin \frac{2\pi L_2}{L} - \sin \frac{\pi L_2}{L})] \end{aligned}$$

$$\begin{aligned}
&= \frac{EI_2}{4} \left(\frac{\pi}{2L} \right)^4 \left[(A^2 + 81B^2) \left(L_2 + \frac{I_1}{I_2} L_1 \right) + A^2 \frac{L}{\pi} \sin \frac{\pi L_2}{L} \left(1 - \frac{I_1}{I_2} \right) + 81B^2 \frac{L}{3\pi} \sin \frac{3\pi L_2}{L} \left(1 - \frac{I_1}{I_2} \right) \right. \\
&\quad \left. + 18AB \frac{L}{\pi} \left\{ \frac{1}{2} \sin \frac{2\pi L_2}{L} \left(1 - \frac{I_1}{I_2} \right) + \sin \frac{\pi L_2}{L} \left(1 - \frac{I_1}{I_2} \right) \right\} \right]. \quad (1.180)
\end{aligned}$$

Defining

$$\alpha = \frac{1}{\pi} \left(1 - \frac{I_1}{I_2} \right), \quad \beta = \frac{EI_2}{4} \left(\frac{\pi}{2L} \right)^4, \quad \gamma = \frac{\pi L_2}{L},$$

we can write Eq. (1.180) as

$$U = \beta \left[(A^2 + 81B^2) \left(L_2 + \frac{I_1}{I_2} L_1 \right) + A^2 L \alpha \sin \gamma + 81B^2 L \frac{\alpha}{3} \sin 3\gamma + 18ABL \alpha \left(\frac{1}{2} \sin 2\gamma + \sin \gamma \right) \right]. \quad (1.181)$$

Now, substituting for dv/dz in Eq. (1.29), we get the potential energy V_e as

$$\begin{aligned}
V_e &= -\frac{P}{2} \int_0^L \left[A^2 \left(\frac{\pi}{2L} \right)^2 \sin^2 \frac{\pi z}{2L} + B^2 \left(\frac{3\pi}{2L} \right)^2 \sin^2 \frac{3\pi z}{2L} + \frac{3}{2} AB \left(\frac{\pi}{2L} \right)^2 \sin \frac{\pi z}{2L} \sin \frac{3\pi z}{2L} \right] dz \\
&= -\frac{P}{4} \left(\frac{\pi}{2L} \right)^2 \int_0^L \left[A^2 \left(1 - \cos \frac{2\pi z}{L} \right) + 9B^2 \left(1 - \cos \frac{6\pi z}{L} \right) + \frac{3}{2} AB \left(\cos \frac{2\pi z}{L} - \cos \frac{4\pi z}{L} \right) \right] dz \\
&= -\frac{P}{4} \left(\frac{\pi}{2L} \right)^2 \left[A^2 \left(z - \sin \frac{\pi z}{L} \frac{L}{\pi} \right) + 9B^2 \left(z - \sin \frac{3\pi z}{L} \frac{L}{3\pi} \right) + \frac{3}{2} AB \left(\sin \frac{\pi z}{L} \frac{L}{\pi} - \sin \frac{2\pi z}{L} \frac{L}{2\pi} \right) \right]_0^L \\
&= -\frac{P}{4} \left(\frac{\pi}{2L} \right)^2 (A^2 + 9B^2) L. \quad (1.183)
\end{aligned}$$

Substituting the expressions for U and V_e in Eq. (1.169) and differentiating with respect to A and B , we get

$$\begin{aligned}
2A \left\{ \left(L_2 + \frac{I_1}{I_2} L \right) + L \alpha \sin \gamma \right\} + 18BL \alpha \left(\frac{1}{2} \sin 2\gamma + \sin \gamma \right) - \frac{P}{2\beta} \left(\frac{\pi}{2L} \right)^2 AL &= 0, \\
162B \left\{ \left(L_2 + \frac{I_1}{I_2} L \right) + L \frac{\alpha}{3} \sin \gamma \right\} + 18AL \alpha \left(\frac{1}{2} \sin 2\gamma + \sin \gamma \right) - \frac{9P}{2\beta} \left(\frac{\pi}{2L} \right)^2 BL &= 0. \quad (1.184)
\end{aligned}$$

Equations (1.184) are a set of simultaneous homogeneous algebraic equations in A and B . Therefore, for a nontrivial solution, the determinant of the matrix formed from the coefficients of A and B must be zero. This leads to the characteristic equation from which P_{cr} can be obtained.

1.8.3 Galerkin's Technique

The Galerkin technique is yet another method for obtaining an approximate buckling load. The main difference between this and the Rayleigh-Ritz method is that it requires the integration of the governing differential equation, whereas the latter requires the integration of the energy expression, of a system.

As already stated, the strain energy and the potential energy for a column are

$$U = \frac{1}{2} \int_0^L EI \left(\frac{d^2 v}{dz^2} \right)^2 dz, \quad (1.22)$$

$$V_e = -\frac{P}{2} \int_0^L \left(\frac{dv}{dz} \right)^2 dz. \quad (1.29)$$

Hence, the total potential Π_p for the column can be written as

$$U + V_e = \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 v}{dz^2} \right)^2 - \frac{P}{2} \left(\frac{dv}{dz} \right)^2 \right] dz. \quad (1.185)$$

Assume that a deformed shape $v = v(z)$ exists for which $\delta(U + V_e) = 0$. The function $v(z)$ must therefore satisfy all the boundary conditions, namely, the kinematic and natural boundary conditions, and must be continuous in the domain. Thus,

$$\delta(U + V_e) = \delta \left[\int_0^L \left\{ \frac{1}{2} EI \left(\frac{d^2 v}{dz^2} \right)^2 - \frac{P}{2} \left(\frac{dv}{dz} \right)^2 \right\} dz \right] = 0. \quad (1.186)$$

Taking the variation of v as δv and assuming that the order of derivatives can be interchanged, we get

$$\int_0^L \left[EI \frac{d^2 v}{dz^2} \frac{d^2}{dz^2} (\delta v) - P \frac{dv}{dz} \frac{d}{dz} (\delta v) \right] dz = 0. \quad (1.187)$$

Integrating Eq. (1.187), we obtain

$$\left[EI \frac{d^2 v}{dz^2} \frac{d}{dz} (\delta v) - EI \frac{d^3 v}{dz^3} \delta v - P \frac{dv}{dz} \delta v \right]_0^L + \int_0^L \left[EI \frac{d^4 v}{dz^4} + P \frac{d^2 v}{dz^2} \right] \delta v \cdot dz = 0. \quad (1.188)$$

Since δv is arbitrary, Eq. (1.188) can be satisfied only if each of its two parts is separately set equal to zero, that is,

$$\int_0^L \left[EI \frac{d^4 v}{dz^4} + P \frac{d^2 v}{dz^2} \right] \delta v \cdot dz = 0, \quad (1.189)$$

$$\left[EI \frac{d^2 v}{dz^2} \frac{d}{dz} (\delta v) - EI \frac{d^3 v}{dz^3} \delta v - P \frac{dv}{dz} \delta v \right]_0^L = 0. \quad (1.190)$$

For Eq. (1.189) to be zero, the integrand must be zero. That is,

$$EI \frac{d^4 v}{dz^4} + P \frac{d^2 v}{dz^2} = 0. \quad (1.191)$$

This is the differential equation of an axially loaded member, obtained earlier (see Section 1.4) from the first principles, and is the same as Eq. (1.48).

Equation (1.190) leads to following possible boundary conditions:

$$EI \frac{d^2 v}{dz^2} = 0 \quad (z = 0, L), \quad (1.192a)$$

$$dv/dz = 0 \quad (z = 0, L), \quad (1.192b)$$

$$EI \frac{d^3 v}{dz^3} + P \frac{dv}{dz} = 0 \quad (z = 0, L), \quad (1.192c)$$

$$v = 0 \quad (z = 0, L). \quad (1.192d)$$

Equation (1.191) can be solved by choosing an approximate series form for v such that each term of the series satisfies all the boundary conditions of the given problem. The advantage in doing this is that the series can be suitably truncated depending on the accuracy desired. Let $\bar{v}$ be the approximating function, defined as

$$\bar{v} = \sum_{i=1}^n a_i f_i(z). \quad (1.193)$$

Since each of the functions $f_i(z)$ satisfies all the boundary conditions, Eq. (1.190) is satisfied exactly when v is replaced by $\bar{v}$. To satisfy Eq. (1.189), the coefficients a_i must be chosen in such a way that $\bar{v}$ satisfies the differential equation (1.191).

Defining the operator $\bar{L}$ as

$$\bar{L} = EI \frac{d^4}{dz^4} + P \frac{d^2}{dz^2}, \quad (1.194)$$

we can write Eq. (1.189) as

$$\int_0^L \bar{L}(\bar{v}) \delta \bar{v} \, dz = 0. \quad (1.195)$$

Since $\bar{v}$ is a function of n parameters, its variation is given by

$$\begin{aligned} \delta \bar{v} &= \frac{\partial \bar{v}}{\partial a_1} \delta a_1 + \frac{\partial \bar{v}}{\partial a_2} \delta a_2 + \dots + \frac{\partial \bar{v}}{\partial a_n} \delta a_n \\ &= \sum_{i=1}^n f_i(z) \delta a_i. \end{aligned} \quad (1.196)$$

Substituting for $\delta \bar{v}$, we can write Eq. (1.195) as

$$\int_0^L \bar{L}(\bar{v}) \sum_{i=1}^n f_i(z) \delta a_i \, dz = 0. \quad (1.197)$$

Since the functions $f_i(z)$ are independent, the only way Eq. (1.197) can be zero is that, for each term, the equation should vanish identically. Thus,

$$\int_0^L \bar{L}(\bar{v}) f_i(z) \delta a_i \, dz = 0 \quad (i = 1, 2, \dots, n).$$

Since the δa_i 's are arbitrary,

$$\int_0^L \bar{L}(\bar{v}) f_i(z) \, dz = 0 \quad (i = 1, 2, \dots, n). \quad (1.198)$$

Equations (1.198) are referred to as the Galerkin equations. For any given problem, these

lead to a set of simultaneous homogeneous algebraic equations, the characteristic equation of which yields the critical load. We may note here that it is rather difficult to choose a displacement function which satisfies both geometric and natural boundary conditions. In view of this, the Rayleigh-Ritz technique is preferred to the Galerkin technique.

We shall illustrate the Galerkin technique through examples.

EXAMPLE 1.6 (Buckling of column with stepped cross-section) Consider the stepped column shown in Fig. 1.16. We have already analyzed it using the Rayleigh-Ritz technique (see Example 1.5). Here, we shall demonstrate the application of the Galerkin technique.

A displacement function of the type

$$\bar{v} = \sum_{i=1,3,5,\dots}^n A_i (1 - \cos \frac{n\pi z}{2L}), \quad i = 1, 3, 5, \dots, \quad (1.199)$$

satisfies the geometric boundary conditions at $z = 0$ and the natural boundary conditions at $z = L$. For simplicity, we shall consider only the first term of the series for $\bar{v}$, namely,

$$\bar{v} = A_1 (1 - \cos \frac{\pi z}{2L}). \quad (1.200)$$

Substituting for $\bar{v}$ in Eqs. (1.198) and replacing $\bar{L}$ in Eqs. (1.198) by Eq. (1.194), we get

$$\begin{aligned} \int_0^{L_1} \{ -EI_2 (\frac{\pi}{2L})^4 \cos \frac{\pi z}{2L} + P_{cr} (\frac{\pi}{2L})^2 \cos \frac{\pi z}{2L} \} (1 - \cos \frac{\pi z}{2L}) dz \\ + \int_{L_1}^L \{ EI_1 (\frac{\pi}{2L})^4 \cos \frac{\pi z}{2L} + P_{cr} (\frac{\pi}{2L})^2 \cos \frac{\pi z}{2L} \} (1 - \cos \frac{\pi z}{2L}) dz = 0. \end{aligned} \quad (1.201)$$

Integrating Eq. (1.201) and substituting the limits for z in the result so obtained, we have

$$P_{cr} = EI_2 \frac{\pi^2}{4L^2} [(\frac{L_2}{L} + \frac{I_1 L_1}{I_2 L}) + \frac{1}{\pi} (1 - \frac{I_1}{I_2}) \sin \frac{\pi L_2}{L} + \frac{4}{\pi} (\frac{I_1}{I_2} - 1) \sin \frac{\pi L_2}{2L}]. \quad (1.202)$$

EXAMPLE 1.7 (Buckling load of clamped-simply-supported column) Consider the column shown in Fig. 1.17. The boundary conditions are

$$v = 0 \quad (z = 0), \quad (1.203)$$

$$dv/dz = 0 \quad (z = 0),$$

$$EI \frac{d^2 v}{dz^2} = 0 \quad (z = L), \quad (1.204)$$

$$v = 0 \quad (z = L).$$

Since it is rather difficult to select a trigonometric function which satisfies these boundary conditions, we choose, instead, a polynomial in z as

$$v = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4. \quad (1.205)$$

In the polynomial so chosen, the number of terms should be one more than the number of boundary conditions

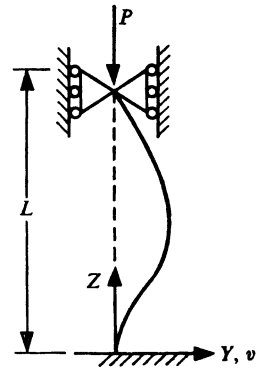


Fig. 1.17 Clamped-Simply-Supported Column.

to be satisfied. The constants (here, $A_0, A_1, \dots, A_4$) can then be determined from the boundary conditions. In doing this, one of the constants, obviously, will remain undetermined.

Substituting Eq. (1.205) in the boundary conditions (1.203) and (1.204), we obtain

$$A_0 = 0, \quad A_1 = 0, \quad A_3 = -\frac{5}{2}A_4L, \quad A_2 = \frac{3}{2}A_4L^2. \quad (1.206)$$

In terms of a new constant, the approximate displacement function $\bar{v}$ can be written as

$$\bar{v} = A(2z^4 - 5Lz^3 + 3L^2z^2). \quad (1.207)$$

Then,

$$\bar{L}(\bar{v}) = A[48EI + P_{cr}(24z^2 - 30Lz + 6L^2)],$$

$$f_1(z) = 2z^4 - 5Lz^3 + 3L^2z^2.$$

Substituting these two expressions in Eqs. (1.198), we get

$$\int_0^L A[48EI(2z^4 - 5Lz^3 + 3L^2z^2) + P_{cr}(48z^6 - 60Lz^5 + 12L^2z^4 - 120Lz^3 + 150L^2z^2 - 30L^3z + 72L^4z^2 - 90L^3z^3 + 18L^4z^2)] dz = 0. \quad (1.208)$$

Integrating Eq. (1.208) and simplifying, we get

$$A(\frac{86}{5}EI - \frac{12}{85}P_{cr}) = 0.$$

Since $A \neq 0$, the critical load is

$$P_{cr} = 21EI/L^2. \quad (1.209)$$

The exact buckling load for the given column is $20.2EI/L^2$ [see Eq. (1.93)].

1.9 NUMERICAL TECHNIQUES

The differential equation approach is useful when it is possible to get a closed-form solution. In the Rayleigh-Ritz and Galerkin techniques, where the approximation function is quite involved and/or contains a large number of terms, it is, in general, difficult to obtain a closed-form solution because the resulting system of algebraic equations is so complicated that a computer is needed to solve it. Since structural modelling is quite complex, it requires the use of comparatively more efficient methods. With the availability of large size and fast computers, the numerical techniques are finding a wider application in this area. Of the available techniques, the finite difference and finite element techniques are well-suited for computer application. (Strictly speaking, the Rayleigh-Ritz and Galerkin techniques should be classified as numerical techniques.) The scope of the former is limited to well-defined boundaries, whereas that of the latter is more general and can therefore be applied to all types of problems. Both the techniques have the ability to converge to the exact solution.

1.9.1 Finite Difference Technique

The finite difference method is a numerical technique for solving differential equations. It essentially reduces a problem having infinite degrees of freedom to one with finite degrees of freedom. This is achieved by substituting an algebraic expression for each unknown func-

tion and its derivative. Since each derivative is replaced by the value of the function at the reference and neighbouring points, the accuracy of the derivatives, and hence of the solution, can be increased if these points are so chosen that they are as close to each other as possible. This technique has the disadvantage that it gives the value of the function only at discrete points. Thus, if an analytical expression has to be obtained for a deflected shape, a curve has then to be fitted passing through all these points. For an extensive discussion on this technique, see Salvadori and Baron [4], Ralston [5], Scarborough [6], Ralston and Wilf [7], and Szilord [8].

For deriving finite difference relations, the Taylor series expansion can be applied if the grid points are evenly-spaced. Depending on the definition of the derivatives at a reference point in terms of the differences of the neighbouring points, a large variety of practical approximations for the derivatives can be obtained (Ralston and Wilf [7]). Notable among these are (i) backward differences, (ii) forward differences, and (iii) central differences. As the name suggests, the backward differences are defined by the points to the left of the reference point. The forward differences, on the other hand, are defined by the points to the right of the reference point. Of (i)–(iii), the central differences are more accurate as the grid points in this are symmetrically located on either side of the reference point. Central differences are particularly useful in the solution of boundary value problems. Figure 1.18 schematically indicates the grid points for central differences.

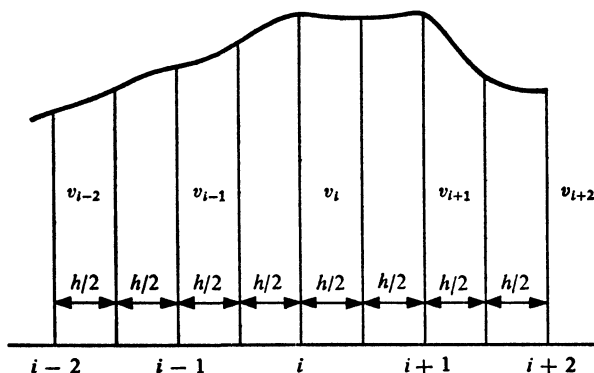


Fig. 1.18 Grid Points for Central Differences.

Let the function $v(z)$ be known at evenly-spaced points and also at the midpoints of the evenly-spaced points. Then, the first central difference of $v(z)$ at i is defined by

$$\begin{aligned} \left(\frac{dv}{dz}\right)_i &\approx \Delta v_i = [v(z_i + \frac{h}{2}) - v(z_i - \frac{h}{2})]/h \\ &= (v_{i+h/2} - v_{i-h/2})/h. \end{aligned} \quad (1.210)$$

This can also be written as

$$\begin{aligned} \left(\frac{dv}{dz}\right)_i &\approx \delta v_i = [v(z_i + h) - v(z_i - h)]/(2h) \\ &= (v_{i+h} - v_{i-h})/(2h). \end{aligned} \quad (1.211)$$

The second difference at i is the difference of the first difference. Hence,

$$\begin{aligned} \left(\frac{d^2v}{dz^2}\right)_i &\approx \delta^2 v_i \equiv \delta(\delta v_i) = [(v_{i+1/2+1/2} - v_{i-1/2+1/2}) - (v_{i+1/2-1/2} - v_{i-1/2-1/2})]/h^2 \\ &= (v_{i+1} - 2v_i - v_{i-1})/h^2. \end{aligned} \quad (1.212)$$

In a similar manner, the third difference can be written as

$$\left(\frac{d^3v}{dz^3}\right)_i \approx \delta^3 v_i = (v_{i+3/2} - 3v_{i+1/2} + 3v_{i-1/2} - v_{i-3/2})/h^3$$

or

$$\left(\frac{d^3v}{dz^3}\right)_i = (v_{i+3} - 3v_{i+1} + 3v_{i-1} - v_{i-3})/(2h^3). \quad (1.213)$$

Further, the fourth difference is given by

$$\left(\frac{d^4v}{dz^4}\right)_i \approx \delta^4 v_i = (v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2})/h^4. \quad (1.214)$$

These difference expressions replace the corresponding derivatives in a governing equation.

We shall now illustrate the application of the technique to two typical problems.

EXAMPLE 1.8 (Buckling of uniform column fixed at one end and hinged at other) The analysis we give here is similar to that presented by Chajes [1].

Consider the uniform column shown in Fig. 1.19. The governing equation is given by (1.59). Further, the boundary conditions in the nondimensional form are given by (1.88),

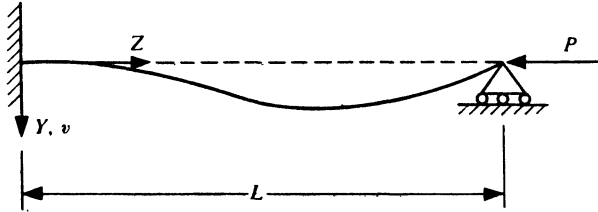


Fig. 1.19 Clamped-Hinged Column under Axial Load.

and (1.89). Dividing the interval $(0, 1)$ of ξ in n equal parts, each of length $h = 1/n$, and substituting Eqs. (1.212) and (1.214) in Eq. (1.59), the finite difference equation at any internal point i we obtain is

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2} + \lambda_n(v_{i+1} - 2v_i + v_{i-1}) = 0, \quad (1.215)$$

where

$$\lambda_n = PL^2/(n^2 EI). \quad (1.216)$$

Equation (1.215) can be rewritten as

$$v_{i+2} + (\lambda_n - 4)v_{i+1} + (6 - 2\lambda_n)v_i + (\lambda_n - 4)v_{i-1} + v_{i-2} = 0. \quad (1.217)$$

Equation (1.217) holds at the $(n - 1)$ internal points $i = 1, 2, \dots, n - 1$. The boundary points of a column are designated as 0 and n . For the boundary conditions, defined at 0 and n , to be satisfied, additional points need to be defined to the left and to the right of the points 0 and n , respectively.

In terms of the central differences, the boundary conditions (1.88) and (1.89) can be written as

$$v_0 = 0, \quad v_{-1} = v_1, \quad v_n = 0, \quad v_{n+1} = -v_{n-1}. \quad (1.218)$$

For a simple support, these mean that the value of the displacement function at a distance h outside the support is approximated by the negative of the functional value at a distance h inside the support. For a fixed support, on the other hand, the value of the displacement function at a distance h outside the support is assumed to be equal to the value of the displacement function at a distance h inside the support. Equations (1.217) lead to a set of $(n - 1)$ linear homogeneous algebraic equations, and have a nontrivial solution only if the determinant of the coefficients of v_i 's is identically zero. It is always advisable to start the solution with a small value of n and then increase the value in stages. This results not only in a saving of the computer storage space and time, but also helps check the convergence of the solution.

First approximation ($n = 2$) The column is divided into two segments, that is, $h = 1/2$. Further, the ends of the segments are denoted as 0, 1, and 2. Writing Eq. (1.217) for the internal point $i = 1$, we get

$$v_3 + (\lambda_2 - 4)v_2 + (6 - 2\lambda_2)v_1 + (\lambda_2 - 4)v_0 + v_{-1} = 0. \quad (1.219)$$

From the boundary conditions (1.218), we have

$$v_0 = 0, \quad v_{-1} = v_1, \quad v_2 = 0, \quad v_3 = -v_1. \quad (1.220)$$

Substituting Eqs. (1.220) in Eq. (1.219), we find the latter reduces to

$$(6 - 2\lambda_2)v_1 = 0. \quad (1.221)$$

Since $v_1 \neq 0$, we should have

$$\lambda_2 = 3. \quad (1.222)$$

Using this value, we get

$$P_{cr} = 12EI/L^2. \quad (1.223)$$

This solution differs from the exact value [given by Eq. (1.93)] by -41 per cent.

Second approximation ($n = 3$) The column is divided into three segments with $h = 1/3$. The reference (i.e., end) points are denoted as 0, 1, 2, and 3. Writing Eq. (1.217) for the internal

points 1, 2, we get

$$\begin{aligned} v_3 + (\lambda_3 - 4)v_2 + (6 - 2\lambda_3)v_1 + (\lambda_3 - 4)v_0 + v_{-1} &= 0, \\ v_4 + (\lambda_3 - 4)v_3 + (6 - 2\lambda_3)v_2 + (\lambda_3 - 4)v_1 + v_0 &= 0. \end{aligned} \quad (1.224)$$

From the boundary conditions (1.218), we have

$$v_0 = 0, \quad v_{-1} = v_1, \quad v_3 = 0, \quad v_4 = -v_2. \quad (1.225)$$

Substituting these in Eqs. (1.224) and simplifying, we obtain the following two linear homogeneous algebraic equations:

$$\begin{aligned} (7 - 2\lambda_3)v_1 + (\lambda_3 - 4)v_2 &= 0, \\ (\lambda_3 - 4)v_1 + (5 - 2\lambda_3)v_2 &= 0. \end{aligned} \quad (1.226)$$

The coefficient matrix of Eqs. (1.226) is given by

$$\begin{bmatrix} 7 - 2\lambda_3 & \lambda_3 - 4 \\ \lambda_3 - 4 & 5 - 2\lambda_3 \end{bmatrix}. \quad (1.227)$$

As can be observed, the matrix is symmetric about the main diagonal. This property can be made use of in minimizing the computer storage space. The characteristic equation corresponding to matrix (1.227) is

$$3\lambda_3^2 - 16\lambda_3 + 19 = 0 \quad (1.228)$$

whose smallest root λ_3 is 1.78. Hence,

$$P_{cr} = (16.603) \frac{EI}{L^2}. \quad (1.229)$$

This solution differs from the exact value [given by Eq. (1.93)] by about -20 per cent.

Third approximation ($n = 4$) The column is divided into four segments. The end points are denoted as 0, 1, 2, 3, and 4. We can now write Eq. (1.217) for the internal points 1, 2, and 3 as

$$\begin{aligned} v_3 + (\lambda_4 - 4)v_2 + (6 - 2\lambda_4)v_1 + (\lambda_4 - 4)v_0 + v_{-1} &= 0, \\ v_4 + (\lambda_4 - 4)v_3 + (6 - 2\lambda_4)v_2 + (\lambda_4 - 4)v_1 + v_0 &= 0, \\ v_5 + (\lambda_4 - 4)v_4 + (6 - 2\lambda_4)v_3 + (\lambda_4 - 4)v_2 + v_1 &= 0. \end{aligned} \quad (1.230)$$

From the boundary conditions (1.218), we have

$$v_0 = 0, \quad v_{-1} = v_1, \quad v_4 = 0, \quad v_5 = -v_3. \quad (1.231)$$

Equations (1.230) then reduce to

$$\begin{aligned} (7 - 2\lambda_4)v_1 + (\lambda_4 - 4)v_2 + v_3 &= 0, \\ (\lambda_4 - 4)v_1 + (6 - 2\lambda_4)v_2 + (\lambda_4 - 4)v_3 &= 0, \\ v_1 + (\lambda_4 - 4)v_2 + (5 - 2\lambda_4)v_3 &= 0. \end{aligned} \quad (1.232)$$

The smallest root of the characteristic equation corresponding to Eqs. (1.232) is $\lambda_4 = 1.111$. Thus,

$$P_{cr} = (17.772) \frac{EI}{L^2}. \quad (1.233)$$

This solution differs from the exact value [given by Eq. (1.93)] by -11.6 per cent.

We can further improve the accuracy by reducing the spacing between the reference points. For this, we shall, instead, use the existing results, i.e., Eqs. (1.223), (1.229), and (1.233), and extrapolate them by applying Richardson's extrapolations (Ralston and Wilf [7]), one of which is

$$(P_{cr})_{n_1, n_2} = \frac{n_2^2}{n_2^2 - n_1^2} (P_{cr})_{n_2} - \frac{n_1^2}{n_2^2 - n_1^2} (P_{cr})_{n_1}, \quad (1.234)$$

where $(P_{cr})_{n_2}$, $(P_{cr})_{n_1}$ are the values at n_2 , n_1 (i.e., the number of segments) and $(P_{cr})_{n_1, n_2}$ is the extrapolated value based on these segments. Let us consider $n_1 = 3$, and $n_2 = 4$. Then,

$$\begin{aligned} (P_{cr})_{3, 4} &= \frac{4^2}{4^2 - 3^2} (P_{cr})_4 - \frac{3^2}{4^2 - 3^2} (P_{cr})_3 \\ &= \frac{16}{7}(17.772) - \frac{9}{7}(16.603) = 19.969. \end{aligned} \quad (1.235)$$

This value differs from the exact value [given by Eq. (1.93)] by -1.1 per cent.

EXAMPLE 1.9 (Buckling of simply-supported column resting on elastic support). Consider a simply-supported column resting on an elastic support of stiffness k per unit length, as

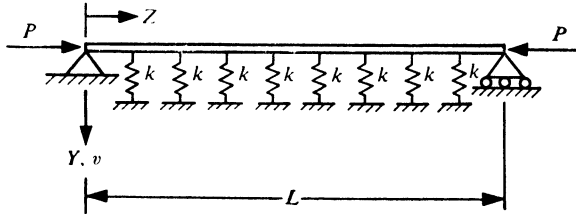


Fig. 1.20 Column on Elastic Support.

shown in Fig. 1.20. The governing equation can be written as

$$EI \frac{d^4 v}{dz^4} + kv + P \frac{d^2 v}{dz^2} = 0, \quad (1.236)$$

where kv is the force due to the elastic support. In terms of the nondimensional parameter ξ , defined as $\xi = z/L$, Eq. (1.236) can be written as

$$\frac{d^4 v}{d\xi^4} + \frac{kL^4}{EI} v + \frac{PL^2}{EI} \frac{d^2 v}{d\xi^2} = 0. \quad (1.237)$$

Dividing the interval (0, 1) of ξ into n equal parts, each of length $h = 1/n$, the difference equation at any internal point i we get is

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2} + K_n v_i + \lambda_n(v_{i+1} - 2v_i + v_{i-1}) = 0, \quad (1.238)$$

where

$$K_n = \frac{kL^4}{n^4 EI}, \quad \lambda_n = \frac{PL^2}{n^2 EI}.$$

Equation (1.238) can be rewritten as

$$v_{i+2} + (\lambda_n - 4)v_{i+1} + (6 + K_n - 2\lambda_n)v_i + (\lambda_n - 4)v_{i-1} + v_{i-2} = 0. \quad (1.239)$$

The boundary conditions are given by (1.4) and (1.5). In terms of the central differences, these give

$$v_0 = 0, \quad v_{-1} = -v_1, \quad v_n = 0, \quad v_{n+1} = -v_{n-1}. \quad (1.240)$$

First approximation ($n = 2$) The column is divided into two segments. The ends of the segments are defined as 0, 1, and 2. For the internal point $i = 1$, Eq. (1.239) can be written as

$$v_3 + (\lambda_2 - 4)v_2 + (6 + K_2 - 2\lambda_2)v_1 + (\lambda_2 - 4)v_0 + v_{-1} = 0. \quad (1.241)$$

Using the boundary conditions (1.240), we can rewrite Eq. (1.241) as

$$(4 + K_2 - 2\lambda_2)v_1 = 0. \quad (1.242)$$

As $v_1 \neq 0$, we have $\lambda_2 = 2 + K_2/2$. Hence,

$$P_{cr} = \left(\frac{8EI}{L^2} + \frac{kL^2}{4} \right). \quad (1.243)$$

Second approximation ($n = 3$) The column is divided into three segments. The ends of the segments are denoted as 0, 1, 2, and 3. The algebraic equations for the internal points 1, 2 are

$$\begin{aligned} v_3 + (\lambda_3 - 4)v_2 + (6 + K_3 - 2\lambda_3)v_1 + (\lambda_3 - 4)v_0 + v_{-1} &= 0, \\ v_4 + (\lambda_3 - 4)v_3 + (6 + K_3 - 2\lambda_3)v_2 + (\lambda_3 - 4)v_1 + v_0 &= 0. \end{aligned} \quad (1.244)$$

Incorporating the boundary conditions (1.240), we obtain

$$\begin{aligned} (5 + K_3 - 2\lambda_3)v_1 + (\lambda_3 - 4)v_2 &= 0, \\ (\lambda_3 - 4)v_1 + (5 + K_3 - 2\lambda_3)v_2 &= 0. \end{aligned} \quad (1.245)$$

The characteristic equation corresponding to Eqs. (1.245) is

$$3\lambda_3^2 - 4(K_3 + 3)\lambda_3 + K_3^2 + 10K_3 + 9 = 0. \quad (1.246)$$

Knowing λ_3 , we can obtain P_{cr} .

1.9.2 Improved Finite Difference Technique

The error in derivatives (1.210)–(1.214) is of the order of h^2 . It is inherent in the derivatives and cannot be eliminated. However, it can be minimized by representing the derivatives by higher order difference expressions where the error is of the order of h^4 . These expressions are given as

$$\Delta v_i = \frac{1}{12h}(v_{i-2} - 8v_{i-1} + 8v_{i+1} - v_{i+2}), \quad (1.247a)$$

$$\Delta^2 v_i = \frac{1}{12h^2}(-v_{i-2} + 16v_{i-1} - 30v_i + 16v_{i+1} - v_{i+2}), \quad (1.247b)$$

$$\Delta^3 v_i = \frac{1}{8h^3}(v_{i-3} - 8v_{i-2} + 13v_{i-1} - 13v_{i+1} + 8v_{i+2} - v_{i+3}), \quad (1.247c)$$

$$\Delta^4 v_i = \frac{1}{6h^4}(-v_{i-3} + 12v_{i-2} - 39v_{i-1} + 56v_i - 39v_{i+1} - 12v_{i+2} - v_{i+3}). \quad (1.247d)$$

For a detailed derivation of these equations, see Szilord [8].

EXAMPLE 1.10 (Buckling of uniform column fixed at one end and hinged at other—Modified scheme) Here, Eq. (1.215) for any internal point i is written as

$$\begin{aligned} & \frac{1}{6}(-v_{i-3} + 12v_{i-2} - 39v_{i-1} + 56v_i - 39v_{i+1} - 12v_{i+2} - v_{i+3}) \\ & + \frac{\lambda_n}{12}(-v_{i-2} + 16v_{i-1} - 30v_i + 16v_{i+1} - v_{i+2}) = 0. \end{aligned} \quad (1.248)$$

Also, the boundary conditions given by (1.88) and (1.89) reduce to

$$\begin{aligned} v_0 &= 0, \\ v_{-2} - 8v_{-1} + 8v_1 - v_2 &= 0, \\ v_n &= 0, \\ -v_{n-2} + 16v_{n-1} - 30v_n + 16v_{n+1} - v_{n+2} &= 0. \end{aligned} \quad (1.249)$$

First approximation ($n = 2$) The column is divided into two segments. The ends of the segments are marked as 0, 1, and 2. Writing Eq. (1.248) for the internal point 1, we get

$$\begin{aligned} & \frac{1}{6}(-v_{-2} + 12v_{-1} - 39v_0 + 56v_1 - 39v_2 + 12v_3 - v_4) \\ & + \frac{\lambda_2}{12}(-v_{-1} + 16v_0 - 30v_1 + 16v_2 - v_3) = 0. \end{aligned} \quad (1.250)$$

From the boundary conditions (1.249), we get

$$\begin{aligned} v_0 &= 0, \\ v_2 - 8v_{-1} + 8v_1 - v_2 &= 0, \\ v_2 &= 0, \\ -v_0 + 16v_1 - 30v_2 + 16v_3 - v_4 &= 0. \end{aligned} \quad (1.251)$$

Making use of relations (1.251) and physical reasonings, we obtain

$$(56 - 15\lambda_2)v_1 = 0. \quad (1.252)$$

Since $v_1 \neq 0$, we should have

$$\lambda_2 = 3.735. \quad (1.253)$$

Using this value, we get

$$P_{cr} = (14.94) \frac{EI}{L^2}. \quad (1.254)$$

Comparing Eqs. (1.254) and (1.223), we find that the former represents an improvement in the solution. Similarly, we can build up higher approximations.

1.9.3 Finite Element Technique

Most numerical techniques lead to solutions that yield approximate values of unknown quantities, e.g., displacement and stiffness, only at selected points in a body. A body can be discretized into an equivalent system of smaller bodies. The assemblage of such bodies (or subsystems) represents the whole body. Each subsystem is solved first and the results so obtained are then combined to obtain a solution for the whole body. Of the numerical techniques, the finite element technique is the most suitable for digital computers. It is applicable to a wide range of problems involving nonhomogeneous materials, nonlinear stress-strain relations, and complicated boundary conditions. Such problems are usually tackled by one of three approaches, namely, (i) the displacement (or stiffness) method, (ii) the equilibrium (or force) method, and (iii) the mixed method. The displacement method, to which we shall confine our discussion, is widely used because of the simplicity with which it can be handled on a computer.

The finite element method was derived on the basis of physical considerations in structural mechanics. A mathematical interpretation of the method was later given by Courant and Hilbert [9]. The credit for making important contributions in developing the technique goes to Turner et al [10] and Argyris and Kelsey [11]. The term "finite element" was first proposed by Clough [12]. For a detailed discussion on the application of this technique to solid and fluid mechanics, see Zienkiewicz [13], Martin and Casey [14], Desai and Abel [15], Gallagher [16], Cook [17], and Bathe and Wilson [18].

In the displacement approach, a structure is divided into a number of finite elements and these elements are interconnected at joints termed nodes. The displacements within each element are then represented by simple functions. The unknown magnitudes of these functions are the displacements or the derivatives of the displacements at the nodes. A displacement function is generally expressed in terms of a polynomial. From the convergence point of view, such a function is so chosen that it (see Desai and Abel [15])

- (i) is continuous within the elements and compatible between adjacent elements,
- (ii) includes the rigid body displacements and rotations of an element,
- (iii) has a constant strain state.

The elements which satisfy conditions (i) and (ii) are termed as *compatible* or *conforming*.

Further, the elements which satisfy conditions (ii) and (iii) are known as *complete*.

When expressing a displacement function in terms of a polynomial, the order of the polynomial has also to be carefully chosen. We shall discuss this in Chapter 5.

The element stiffness matrix is calculated by applying one of the variational principles of solid mechanics. The total potential Π_p for an element can be written as

$$\begin{aligned} (\Pi_p)_e &= \text{strain energy } (U) + \text{potential energy of applied force } (V_e) \text{ (or negative of} \\ &\quad \text{work done)} \\ &= \frac{1}{2} \int_{\text{vol}} \epsilon^T \sigma \, dv - \int_{S_1} (\bar{\phi}_x u + \bar{\phi}_y v + \bar{\phi}_z w) \, dS_1 - \int_{\text{vol}} (\bar{X}_x u + \bar{Y}_y v + \bar{Z}_z w) \, dv, \end{aligned} \quad (1.255)$$

where S_1 is the part of the surface of the body on which the surface forces $\bar{\phi}_x$, $\bar{\phi}_y$, $\bar{\phi}_z$ are imposed; $\bar{X}_x$, $\bar{Y}_y$, $\bar{Z}_z$ are the prescribed body forces per unit volume; u , v , w are the displacements along the X -, Y -, Z -direction; and σ , ϵ represent all the six independent components of stress and strain for a three-dimensional body. Because the integral of the summation is the same as the sum of the individual integrals, the total potential can be written as

$$\begin{aligned} \Pi_p &= \frac{1}{2} \sum_{e=1}^N \int_{\text{vol}} \epsilon^T \sigma \, dv - \sum_{e=1}^N \int_{S_1} (\bar{\phi}_x u + \bar{\phi}_y v + \bar{\phi}_z w) \, dS_1 \\ &\quad - \sum_{e=1}^N \int_{\text{vol}} (\bar{X}_x u + \bar{Y}_y v + \bar{Z}_z w) \, dv, \end{aligned} \quad (1.256)$$

where e is the index denoting an element and N is the total number of elements of the structure.

For a one-dimensional structure, a displacement $u(x)$ within an element can be approximated by a polynomial as

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots + \alpha_{n+1} x^n, \quad (1.257)$$

where the coefficients $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ are called the *generalized coordinates*. These are so named because they do not represent any physical quantity, e.g., displacement and rotation, on a one-to-one basis. In general, they are linear combinations of some of the nodal displacements and the derivatives of these displacements. The larger the value n , the greater the accuracy with which the displacement is approximated. In matrix notation, Eq. (1.257) can be written as

$$u(x) = \phi^T \alpha, \quad (1.258)$$

where

$$\begin{aligned} \phi^T &= [1 \quad x \quad x^2 \quad \dots \quad x^n], \\ \alpha^T &= [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{n+1}]. \end{aligned} \quad (1.259)$$

The relation between the nodal degrees of freedom $q_1, q_2, \dots, q_M$ and the generalized coordinates $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ can be expressed as

$$u = [\phi] \alpha. \quad (1.260)$$

If we denote $q_1 = u$ (at node 1), $q_2 = u$ (at node 2), and so on, then

$$\mathbf{q} = \begin{Bmatrix} u_{\text{node 1}} \\ u_{\text{node 2}} \\ \vdots \\ u_{\text{node } M} \end{Bmatrix} = \begin{Bmatrix} [\phi]_{\text{node 1}} \\ [\phi]_{\text{node 2}} \\ \vdots \\ [\phi]_{\text{node } M} \end{Bmatrix} \boldsymbol{\alpha} = [\mathbf{A}] \boldsymbol{\alpha}, \quad (1.261)$$

where M is the total number of nodes for an element and $\mathbf{q}$ is the vector of nodal displacements. Substituting for $\boldsymbol{\alpha}$ from Eq. (1.260) in Eq. (1.261), we have

$$\begin{aligned} u &= [\phi][\mathbf{A}]^{-1} \mathbf{q} \\ &= [\mathbf{N}] \mathbf{q}. \end{aligned} \quad (1.262)$$

Equation (1.262) expresses the displacement at any point in terms of the nodal displacements. To obtain the stiffness matrix from the total potential, we need an expression for stress and strain in terms of the displacement. The strain-displacement relation is given by

$$\boldsymbol{\epsilon} = [\mathbf{B}] \mathbf{q} \quad (1.263a)$$

or

$$\boldsymbol{\epsilon} = [\mathbf{B}_e] \boldsymbol{\alpha}. \quad (1.263b)$$

The matrices $[\mathbf{B}]$ and $[\mathbf{B}_e]$ are related as

$$[\mathbf{B}] = [\mathbf{B}_e][\mathbf{A}]^{-1}.$$

If the material is linearly elastic, the stress and strain are related as

$$\boldsymbol{\sigma} = [\mathbf{X}] \boldsymbol{\epsilon}, \quad (1.264)$$

where $[\mathbf{X}]$ is the material matrix. Substituting for $\boldsymbol{\epsilon}$ from Eq. (1.263a) or (1.263b), we have

$$\boldsymbol{\sigma} = [\mathbf{X}][\mathbf{B}_e] \boldsymbol{\alpha} \quad (1.265a)$$

$$= [\mathbf{X}][\mathbf{B}] \mathbf{q}. \quad (1.265b)$$

Now, substituting Eqs. (1.262), (1.264), and (1.265b) in Eq. (1.255), we obtain

$$(\Pi_p)_e = \frac{1}{2} \int_{\text{vol}} \mathbf{q}^T [\mathbf{B}]^T [\mathbf{X}] [\mathbf{B}] \mathbf{q} \, dv - \int_{S_1} \mathbf{q}^T [\mathbf{N}]^T \bar{\boldsymbol{\phi}} \, dS_1 - \int_{\text{vol}} \mathbf{q}^T [\mathbf{N}]^T \bar{\mathbf{X}} \, dv. \quad (1.266)$$

Applying the variational principle $\delta \Pi_p = 0$, we get

$$\delta \mathbf{q}^T \left[\int_{\text{vol}} [\mathbf{B}]^T [\mathbf{X}] [\mathbf{B}] \mathbf{q} \, dv - \int_{\text{vol}} [\mathbf{N}]^T \bar{\mathbf{X}} - \int_{S_1} [\mathbf{N}]^T \bar{\boldsymbol{\phi}} \, dS_1 \right] = 0. \quad (1.267)$$

Since $\delta \mathbf{q}^T$ is arbitrary, for Eq. (1.267) to be satisfied, the expression within the square brackets must be zero. This gives the equilibrium equation for an element as

$$[\mathbf{k}] \mathbf{q} = \mathbf{Q}_L. \quad (1.268)$$

where $[k]$ is the elemental stiffness matrix given by

$$[k] = \int_{\text{vol}} [B]^T [X] [B] dv \quad (1.269)$$

and $\mathbf{Q}_L$ is the external load vector written as

$$\mathbf{Q}_L = \int_{\text{vol}} [N]^T \bar{X} dv + \int_{S_1} [N]^T \bar{\phi} dS_1. \quad (1.270)$$

As stated in Section 1.2.2, a stability problem can be formulated by using the principle of stationary total potential

$$\delta(U - W_d) = 0, \quad (1.271)$$

where U is the strain energy and W_d is the work done by the inplane loads. In terms of finite elements, U and W_d can be written as

$$\begin{aligned} U &= \frac{1}{2} \mathbf{q}^T [K] \mathbf{q}, \\ W_d &= \frac{\lambda}{2} \mathbf{q}^T [G] \mathbf{q}, \end{aligned} \quad (1.272)$$

where $[K]$ and $[G]$ are the elastic stiffness matrix and the geometric stiffness matrix, respectively, for the structure, $\mathbf{q}$ the vector of global displacements, and λ the eigenvalue. Equation (1.269) gives the elastic stiffness matrix for an element. The total stiffness matrix can be obtained by adding the element stiffness matrices.

To obtain $[G]$, consider an element subjected to the inplane stresses σ_x , σ_y , and τ_{xy} . The work done by these inplane stresses due to lateral deformation of the element is given by

$$W_d = \frac{1}{2} \sigma_x \int_{\text{vol}} \left(\frac{\partial u}{\partial x} \right)^T \left(\frac{\partial u}{\partial x} \right) dv + \frac{1}{2} \sigma_y \int_{\text{vol}} \left(\frac{\partial u}{\partial y} \right)^T \left(\frac{\partial u}{\partial y} \right) dv + \tau_{xy} \int_{\text{vol}} \left(\frac{\partial u}{\partial x} \right)^T \left(\frac{\partial u}{\partial y} \right) dv. \quad (1.273)$$

Using Eq. (1.260), i.e., $u = [\phi] \alpha$, we obtain

$$\begin{aligned} W_d &= \frac{1}{2} \sigma_x \alpha^T \left[\int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial x} \right) dv \right] \alpha + \frac{1}{2} \sigma_y \alpha^T \left[\int_{\text{vol}} \left(\frac{\partial \phi}{\partial y} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv \right] \alpha \\ &\quad + \tau_{xy} \alpha^T \left[\int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv \right] \alpha. \end{aligned} \quad (1.274)$$

Taking the variation with respect to α , we get

$$(\delta \alpha)^T \left[\sigma_x \int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial x} \right) dv + \sigma_y \int_{\text{vol}} \left(\frac{\partial \phi}{\partial y} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv + 2 \tau_{xy} \int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv \right]. \quad (1.275)$$

The geometric stiffness matrix $[g_\alpha]$ is given by

$$[g_\alpha] = \sigma_x \int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial x} \right) dv + \sigma_y \int_{\text{vol}} \left(\frac{\partial \phi}{\partial y} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv + 2 \tau_{xy} \int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv. \quad (1.276)$$

The geometric stiffness matrix $[g]$ in terms of the nodal displacements $\mathbf{q}$ is obtained, using

Eq. (1.261), as

$$[g] = [A^{-1}]^T [g_a] [A^{-1}]. \quad (1.277)$$

Knowing the element geometric stiffness matrix $[g]$, we can obtain the total geometric stiffness matrix $[G]$. There are two methods for obtaining the geometric stiffness matrix $[g]$, namely, (i) the consistent geometric stiffness technique and (ii) the nonconsistent geometric stiffness technique. In the former, the same displacement function is used to derive both the elastic stiffness matrix and the geometric stiffness matrix, whereas, in the latter, a lower order displacement function is employed to get the geometric stiffness matrix. The main advantage of the consistent geometric stiffness method is that the eigenvalues are more accurate and are proven upper bounds to the exact solution (Zienkiewicz [13]). In the non-consistent stiffness approach, it is not possible to say *a priori* how the solution will converge.

EXAMPLE 1.11 (Buckling of uniform column) Here, we shall first derive the elastic stiffness and geometric stiffness matrices for a uniform column under axial compression and then obtain these matrices for specified boundary conditions.

Figure 1.21 shows a uniform column. It is divided into two elements of length l ; each element has two degrees of freedom, corresponding to translation and rotation, at each node. Thus, there are four degrees of freedom for each element. Accordingly, we take the displacement function as

$$v = \alpha_1 + \alpha_2 z + \alpha_3 z^2 + \alpha_4 z^3. \quad (1.278)$$

This function satisfies the conditions

of constant shear and linearly varying bending moment that exist in a beam element. If v_1 , v_3 and v_2 , v_4 represent the nodal displacements and rotations, respectively, then the displacement v at any point within an element is given by

$$v = \left[\left(-\frac{2z^3}{l^3} + \frac{3z^2}{l^2} + 1 \right) \left(z - \frac{2z^2}{l} + \frac{z^3}{l^2} \right) \left(-\frac{3z^2}{l^2} + \frac{2z^3}{l^3} \right) \left(\frac{z^3}{l^2} - \frac{z^2}{l} \right) \right] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}. \quad (1.279)$$

Hence,

$$\epsilon_z = -\gamma \frac{\partial^2 v}{\partial z^2} = -\gamma \left[\left(-\frac{12z}{l^3} + \frac{6}{l^2} \right) \left(\frac{6z}{l^2} - \frac{4}{l} \right) \left(-\frac{6}{l^2} + \frac{12z}{l^3} \right) \left(\frac{6z}{l^2} - \frac{2}{l} \right) \right] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}. \quad (1.280)$$

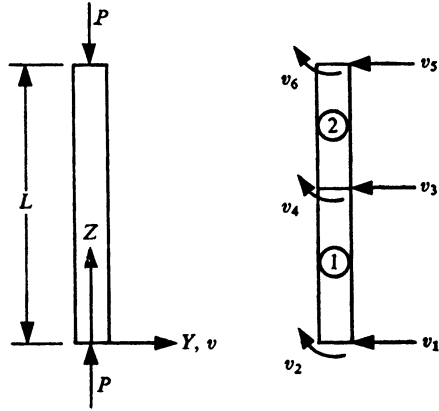


Fig. 1.21 Uniform Column and Its Finite Element Idealization.

Since this is the only strain component present, comparing Eq. (1.270) with Eq. (1.263a), we get

$$[B] = -\gamma \left[\left(-\frac{12z}{l^3} + \frac{6}{l^2} \right) \left(\frac{6z}{l^2} - \frac{4}{l} \right) \left(-\frac{6}{l^2} + \frac{12z}{l^3} \right) \left(\frac{6z}{l^2} - \frac{2}{l} \right) \right]. \quad (1.281)$$

From Eq. (1.269), we have

$$[k] = \int_{\text{vol}} [B]^T [X] [B] dA dz.$$

For a linear, elastic, isotropic material, $[X]$ is given by the elastic modulus E . Substituting the expression for $[B]$ in Eq. (1.269) and integrating, we get

$$[k] = EI \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ 12/l^3 & -6/l^2 & -12/l^3 & -6/l^2 \\ -6/l^2 & 4/l & 6/l^2 & 2/l \\ -12/l^3 & 6/l^2 & 12/l^3 & 6/l^2 \\ -6/l^2 & 2/l & 6/l^2 & 4/l \end{bmatrix}. \quad (1.282)$$

Similarly, we can obtain the consistent geometric stiffness matrix $[g]$ as

$$[g] = P \begin{bmatrix} 6/5l & -1/10 & -6/5l & -1/10 \\ -1/10 & 2l/15 & 1/10 & -l/30 \\ -6/5l & 1/10 & 6/5l & 1/10 \\ -1/10 & -l/30 & 1/10 & 2l/15 \end{bmatrix}. \quad (1.283)$$

Matrices (1.282) and (1.283) are for a single element. For the given column which is divided into two elements, the nodal displacement vector is given by $[v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6]^T$, v_1, v_3, v_5 representing the displacements and v_2, v_4, v_6 the rotations.

To obtain the elastic stiffness matrix for the complete structure, the element stiffness matrices are superimposed. This is done by writing each element matrix for all the degrees of freedom and adding the matrices element-wise. Thus,

$$[k]_1 = \frac{EI}{l^3} \begin{bmatrix} v_1 & v_2l & v_3 & v_4l & v_5 & v_6l \\ 12 & -6 & -12 & -6 & 0 & 0 \\ -6 & 4 & 6 & 2 & 0 & 0 \\ -12 & 6 & 12 & 6 & 0 & 0 \\ -6 & 2 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.284)$$

$$[k]_2 = \frac{EI}{l^3} \begin{bmatrix} v_1 & v_2 l & v_3 & v_4 l & v_5 & v_6 l \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & -6 & -12 & -6 \\ 0 & 0 & -6 & 4 & 6 & 2 \\ 0 & 0 & -12 & 6 & 12 & 6 \\ 0 & 0 & -6 & 2 & 6 & 4 \end{bmatrix}. \quad (1.285)$$

Combining Eqs. (1.284) and (1.285), the total elastic stiffness matrix we obtain is

$$[K] = [k]_1 + [k]_2$$

$$= \frac{EI}{l^3} \begin{bmatrix} v_1 & v_2 l & v_3 & v_4 l & v_5 & v_6 l \\ 12 & -6 & -12 & -6 & 0 & 0 \\ -6 & 4 & 6 & 2 & 0 & 0 \\ -12 & 6 & 24 & 0 & -12 & -6 \\ -6 & 2 & 0 & 8 & 6 & 2 \\ 0 & 0 & -12 & 6 & 12 & 6 \\ 9 & 0 & -6 & 2 & 6 & 4 \end{bmatrix} \quad (1.286)$$

In a similar manner, the total geometric stiffness matrix can be written as

$$[G] = -\frac{P}{l} \begin{bmatrix} v_1 & v_2 l & v_3 & v_4 l & v_5 & v_6 l \\ 6/5 & -1/10 & -6/5 & -1/10 & 0 & 0 \\ -1/10 & 2/15 & 1/10 & -1/30 & 0 & 0 \\ -6/5 & 1/10 & 12/5 & 0 & -6/5 & -1/10 \\ -1/10 & -1/30 & 0 & 4/15 & 1/10 & -1/30 \\ 0 & 0 & -6/5 & 1/10 & 6/5 & 1/10 \\ 0 & 0 & -1/10 & -1/30 & 1/10 & 2/15 \end{bmatrix}. \quad (1.287)$$

The governing equation for buckling is

$$[[K] - \lambda[G]]v = 0. \quad (1.288)$$

Since v is not zero, for a nontrivial solution, the determinant

$$|[K] - \lambda[G]| = 0.$$

We shall now derive the stiffness matrices for specific boundary conditions.

(i) *Simply-supported column* For the first mode of buckling, which is symmetric, $v_1 = v_4 = v_5 = 0$. Further, we can consider only one half of the column as the column is

symmetric. The elastic and geometric stiffness matrices reduce to

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 4 & 6 \\ 6 & 24 \end{bmatrix}, \quad [G] = -\frac{P}{l} \begin{bmatrix} 2/15 & 1/10 \\ 1/10 & 12/5 \end{bmatrix}. \quad (1.289)$$

Therefore,

$$|[K] - \lambda[G]| = \begin{vmatrix} \frac{4EI}{l^3} - \frac{2P}{15l} & \frac{6EI}{l^3} - \frac{P}{10l} \\ \frac{6EI}{l^3} - \frac{P}{10l} & \frac{24EI}{l^3} - \frac{12P}{5l} \end{vmatrix} = 0. \quad (1.290)$$

The lowest value of $P = P_{cr}$ is then

$$P_{cr} = \frac{2.45EI}{l^2} = \frac{9.8EI}{L^2},$$

$$P_{cr}(\text{exact}) = \frac{\pi^2 EI}{L^2} = \frac{9.87EI}{L^2}. \quad (1.291)$$

(ii) *Clamped-clamped column* From the boundary conditions, $v_1 = v_2 = v_4 = v_5 = v_6 = 0$. Once again, considering only one half of the column, the characteristic equation we obtain is

$$24EI/l^3 - 12P_{cr}/(5l) = 0 \quad (1.292)$$

or

$$P_{cr} = 10EI/l^2 = 40EI/L^2, \quad (1.293)$$

$$P_{cr}(\text{exact}) = 4\pi^2 EI/L^2. \quad (1.294)$$

EXAMPLE 1.12 (Buckling of tapered simply-supported column) For a linearly tapered column of rectangular cross-section,

$$I(\xi) = I_0 + (I_1 - I_0)\frac{z}{l}, \quad (1.295)$$

where I_0 is the second moment of area at $z = 0$ and α is the taper parameter. Equation (1.295) is also true for a column of rectangular cross-section with constant breadth and linear depth taper (Fig. 1.22). Such a column can be divided into a number of elements depending

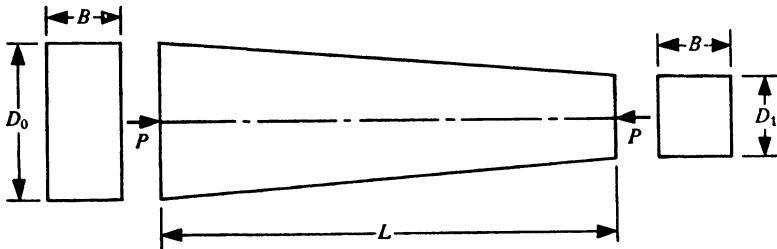


Fig. 1.22 Tapered Beam of Rectangular Cross-Section under Axial Compressive Load.

on the desired accuracy and also on the mode of buckling. We shall, however, assume that it is divided into two elements, and derive the elastic stiffness and geometric stiffness matrices, using both the consistent and nonconsistent approaches.

(i) *Consistent approach* The column is divided into two elements 1 and 2 (see Fig. 1.23). This figure also shows the nodal degrees of freedom.

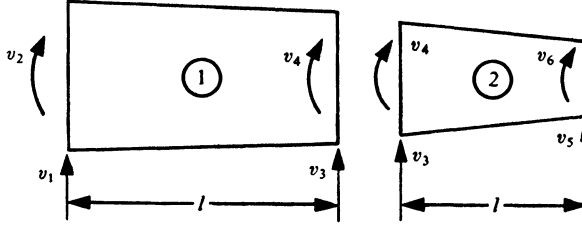


Fig. 1.23 Element Discretization with Nodal Degrees of Freedom.

Let us consider the cubic displacement function

$$v = \alpha_1 + \alpha_2 z + \alpha_3 z^2 + \alpha_4 z^3 \quad (1.278)$$

over an element and derive the element matrices $[k]$ and $[g]$. The generalized coordinates $\alpha_1, \alpha_2, \alpha_3$, and α_4 can be obtained in terms of the nodal degrees of freedom v_1, v_2, v_3 , and v_4 . Further, the matrix $[B]$ for an element is given by (1.281). The elastic stiffness matrix can therefore be obtained by using Eq. (1.269). As already noted, the geometric stiffness matrix for an element is

$$[g] = \sigma_x \int_{\text{vol}} \left[\frac{\partial N}{\partial z} \right]^T \left[\frac{\partial N}{\partial z} \right] dv. \quad (1.296)$$

Integrating Eq. (1.296) over the cross-sectional area, we get

$$[g] = P \int_0^l \left[\frac{\partial N}{\partial z} \right]^T \left[\frac{\partial N}{\partial z} \right] dz. \quad (1.297)$$

For element 1,

$$\begin{aligned} I(z) &= I_0 + (I_1 - I_0) \frac{z}{L} \\ &= I_0 + (I_1 - I_0) \frac{z}{2l}. \end{aligned} \quad (1.298)$$

In terms of the nondimensional parameter ξ ($z = l\xi$), we find Eq. (1.298) becomes

$$I(\xi) = I_0 + (I_1 - I_0) \frac{\xi}{2}. \quad (1.299)$$

From Eq. (1.269), we have

$$\begin{aligned} [k]_1 &= \int_{\text{vol}} [B]^T [X] [B] dv \\ &= \int_0^1 \int_A [B]^T E [B] I dA_c d\xi. \end{aligned} \quad (1.300)$$

Substituting for $[B]$ from Eq. (1.281) in Eq. (1.300) and integrating with respect to the area A_c , we get

$$\begin{aligned} [k]_1 &= \frac{E}{I^3} \int_0^1 [kk] I(\xi) d\xi, \\ [kk] &= \begin{bmatrix} 36(1-4\xi+4\xi^2) & 12(-6\xi^2+7\xi-2)I & 4(4-12\xi+9\xi^2)I^2 & \text{Symmetric} \\ -36(1-4\xi+4\xi^2) & -12(-6\xi^2+7\xi-2)I & 36(1-4\xi+4\xi^2) & \\ 12(-1+5\xi-6\xi^2)I & 4(2-9\xi+9\xi^2)I^2 & -12(-1+5\xi-6\xi^2)I & 4(1-6\xi+9\xi^2)I^2 \end{bmatrix}. \end{aligned} \quad (1.301)$$

Substituting for $I(\xi)$ from Eq. (1.299) in Eq. (1.301) and integrating, we obtain

$$[k]_1 = \frac{E}{I^3} \begin{bmatrix} 9I_0 + 3I_1 & & \text{Symmetric} & \\ -I(5I_0 + I_1) & \frac{I^2}{2}(7I_0 + I_1) & & \\ -9I_0 - 3I_1 & I(5I_0 + I_1) & 9I_0 + 3I_1 & \\ -I(4I_0 + 2I_1) & \frac{I^2}{2}(3I_0 + I_1) & I(4I_0 + 2I_1) & \frac{I^2}{2}(5I_0 + 3I_1) \end{bmatrix}. \quad (1.302)$$

If we use the same representation for $I(\xi)$ as in Eq. (1.299), then ξ varies from 1 to 2. We therefore rewrite $I(\xi)$ such that ξ once again is defined within the limits 0 to 1. $I(\xi)$ is then given by

$$I(\xi) = \frac{I_1 + I_0}{2} + (I_1 - \frac{I_1 + I_0}{2})\xi. \quad (1.303)$$

Substituting for $I(\xi)$ from expression (1.303) in Eq. (1.301) and integrating, we get, for element 2,

$$[k]_2 = \frac{E}{I} \begin{bmatrix} 3I_0 + 9I_1 & & \text{Symmetric} & \\ -I(2I_0 + 4I_1) & \frac{I^2}{2}(3I_0 + 5I_1) & & \\ -3I_0 - 9I_1 & I(2I_0 + 4I_1) & 3I_0 + 9I_1 & \\ -I(I_0 + 5I_1) & \frac{I^2}{2}(I_0 + 3I_1) & I(I_0 + 5I_1) & \frac{I^2}{2}(I_0 + 7I_1) \end{bmatrix}. \quad (1.304)$$

As the local coordinates and the global coordinates are the same, the sum $[k]_1 + [k]_2$ gives the global elastic stiffness matrix for the total structure as

$$[K] = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 9I_0 + 3I_1 & & & & & \\ -l(5I_0 + I_1) & \frac{l^3}{2}(7I_0 + I_1) & & & & \\ -9I_0 - 3I_1 & l(5I_0 + I_1) & 12I_0 + 12I_1 & & & \\ -l(4I_0 + 2I_1) & \frac{l^3}{2}(3I_0 + I_1) & l(2I_0 - 2I_1) & \frac{l^3}{2}(8I_0 + 8I_1) & & \\ 0 & 0 & -3I_0 - 9I_1 & l(2I_0 + 4I_1) & 3I_0 + 9I_1 & \\ 0 & 0 & -l(I_0 + 5I_1) & \frac{l^3}{2}(I_0 + 3I_1) & l(I_0 + 5I_1) & \frac{l^3}{2}(I_0 + 7I_1) \end{bmatrix} \quad (1.305)$$

The geometric stiffness matrix will be the same as given by Eq. (1.287). For our example, it can be written as

$$[G] = P \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 6/5l & -1/10 & -6/5l & -1/10 & 0 & 0 \\ -1/10 & 2l/15 & 1/10 & -l/30 & 0 & 0 \\ -6/5l & 1/10 & 12/5l & 0 & -6/5l & -1/10 \\ -1/10 & -l/30 & 0 & 4l/15 & 1/10 & -l/30 \\ 0 & 0 & -6/5l & 1/10 & 6/5l & 1/10 \\ 0 & 0 & -1/10 & -l/30 & 1/10 & 2l/15 \end{bmatrix} \quad (1.306)$$

We shall now consider the case $I_0/I_1 = 1/2$. Substituting the boundary conditions $v_1 = v_5 = 0$, we find the elastic stiffness matrix $[K]$ and the geometric stiffness matrix $[G]$ reduce to

$$[K] = \frac{EI_0}{l^3} \begin{bmatrix} v_2 & v_3 & v_4 & v_6 \\ \frac{13}{4}l^2 & \frac{11}{2}l & \frac{7}{4}l^2 & 0 \\ \frac{11}{2}l & 18 & l & -\frac{7}{2}l \\ \frac{7}{4}l^2 & l & 6l^2 & \frac{5}{4}l^2 \\ 0 & -\frac{7}{2}l & \frac{5}{4}l^2 & \frac{3}{4}l^2 \end{bmatrix}, \quad (1.307)$$

$$[G] = P \begin{bmatrix} v_2 & v_3 & v_4 & v_6 \\ \frac{2l}{15} & \frac{1}{10} & -\frac{l}{30} & 0 \\ \frac{1}{10} & \frac{2l}{15} & 0 & -\frac{1}{10} \\ -\frac{l}{30} & 0 & \frac{4l}{15} & -\frac{l}{30} \\ 0 & -\frac{1}{10} & -\frac{l}{30} & \frac{2l}{15} \end{bmatrix} \quad (1.308)$$

Substituting for $[K]$ and $[G]$ in Eq. (1.288), we see that $[[K] - \lambda[G]] = 0$ becomes

$$\begin{bmatrix} 15l^2 - 4l^2\lambda & 22l - 3l\lambda & 7l^2 + l^2\lambda & 0 \\ 22l - 3l\lambda & 72 - 72\lambda & 4l & -14l + 3l\lambda \\ 7l^2 + l^2\lambda & 4l & 24l^2 - 8l^2\lambda & 5l^2 + l^2\lambda \\ 0 & -14l + 3l\lambda & 5l^2 + l^2\lambda & 9l^2 - 4l^2\lambda \end{bmatrix} = 0, \quad (1.309)$$

where $\lambda = 2Pl^2/(15EI_0)$. It can be seen that the characteristic equation is a quartic in λ . Its lowest root corresponds to the first buckling load. This root, when evaluated numerically, has the value

$$\lambda_{\min} = 0.2448. \quad (1.310)$$

This corresponds to

$$P_{cr} = (1.836) \frac{EI_0}{l^2} = (7.344) \frac{EI_0}{L^2}. \quad (1.311)$$

(ii) *Nonconsistent approach* Here, the elastic stiffness matrix is obtained by considering a cubic function for displacement. However, the geometric stiffness matrix is derived by choosing a linear shape function. This implies that we have to neglect the rotational degrees of freedom. In view of these, the elastic stiffness matrix will be of the order $[6 \times 6]$, whereas the geometric stiffness matrix will be of the order $[3 \times 3]$. For the two matrix equations to be compatible, we need to calculate $[K_{eff}]$ which is defined as (Desai and Abel [15])

$$[K] = [K_{eff}] = [K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}], \quad (1.312)$$

where $[K_{11}]$, $[K_{12}]$, ... are obtained after partitioning the stiffness matrix. The stiffness matrix is the same as given by (1.307), i.e.,

$$[K] = \frac{EI_0}{l^3} \begin{bmatrix} 18 & \frac{11}{2}l & l & -\frac{7}{2}l \\ \frac{11}{2}l & \frac{15}{4}l^2 & \frac{7}{4}l^2 & 0 \\ l & \frac{7}{4}l^2 & 6l^2 & \frac{5}{4}l^2 \\ -\frac{7}{2}l & 0 & \frac{5}{4}l^2 & \frac{9}{4}l^2 \end{bmatrix} \quad (1.307)$$

Thus,

$$\begin{aligned} [K_{11}] &= [18], \\ [K_{12}] &= [\frac{11}{2}l \quad l \quad -\frac{7}{2}l], \\ [K_{21}] &= [\frac{11}{2}l \quad l \quad -\frac{7}{2}l]^T, \\ [K_{22}] &= \begin{bmatrix} \frac{15}{4}l^2 & \frac{7}{4}l^2 & 0 \\ \frac{7}{4}l^2 & 6l^2 & \frac{5}{4}l^2 \\ 0 & \frac{5}{4}l^2 & \frac{9}{4}l^2 \end{bmatrix}. \end{aligned} \quad (1.313)$$

Substituting for $[K_{11}]$, $\dots$, $[K_{22}]$ in Eq. (1.312), we get $[K] = (4.457)EI/L^3$. This corresponds to the degree of freedom v_3 since each of v_1 and v_5 is zero.

The geometric stiffness matrix is calculated by considering only two transverse degrees of freedom for each element. Keeping the same notation as before, for the two elements these will be v_1 , v_3 , and v_5 . The displacement v at any point within element 1 (see Fig. 1.22) is given by

$$v = [1 - z/l, \quad z/l] \begin{Bmatrix} v_1 \\ v_3 \end{Bmatrix}. \quad (1.314)$$

Following the same steps as given in Section 1.9.3, we find the geometric stiffness matrix for element 1 is

$$[g]_1 = \frac{P}{l} \begin{bmatrix} & v_1 & v_3 & v_5 \\ & 1 & -1 & 0 \\ -1 & & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (1.315)$$

Similarly, for element 2, we get

$$[g]_2 = \frac{P}{l} \begin{bmatrix} & v_1 & v_3 & v_5 \\ & 0 & 0 & 0 \\ 0 & & 1 & -1 \\ 0 & -1 & & 1 \end{bmatrix}. \quad (1.316)$$

The geometric stiffness matrix for the total structure is then given by

$$\begin{aligned} [G] &= [g]_1 + [g]_2 \\ &= \frac{P}{l} \begin{bmatrix} & v_1 & v_3 & v_5 \\ & 1 & -1 & 0 \\ -1 & & 2 & -1 \\ 0 & -1 & & 1 \end{bmatrix}. \end{aligned} \quad (1.317)$$

For the given structure $v_1 = v_5 = 0$. Making use of this, we get P_{cr} as

$$P_{cr} = (8.913) \frac{EI_0}{L^2}. \quad (1.318)$$

Comparing the two values given by Eqs. (1.311) and (1.318), we find that they differ by 20 per cent.

EXAMPLE 1.13 (Buckling load of tapered cantilever column) Consider a column whose second moment of area is as given by Eq. (1.295). As in Example 1.12, so too here we assume that the column is divided into two elements with v_1 , v_3 , v_5 representing the nodal displacements and v_2 , v_4 , v_6 the nodal rotations.

(i) *Consistent approach* The boundary conditions are

$$v_1 = v_2 = 0. \quad (1.319)$$

Incorporating these in Eqs. (1.305) and (1.306), the elastic stiffness and geometric stiffness matrices for this column we obtain are

$$[K] = \frac{EI_0}{l^3} \begin{bmatrix} v_3 & v_4 & v_5 & v_6 \\ 18 & l & -\frac{15}{2} & -\frac{7}{2}l \\ l & 6l^2 & 4l & \frac{5}{4}l^2 \\ -\frac{15}{2} & 4l & \frac{15}{2} & \frac{7}{2}l \\ -\frac{7}{2}l & \frac{5}{4}l^2 & \frac{7}{2}l & \frac{9}{4}l^2 \end{bmatrix}, \quad (1.320)$$

$$[G] = P \begin{bmatrix} v_3 & v_4 & v_5 & v_6 \\ \frac{12}{5l} & 0 & -\frac{6}{5l} & -\frac{1}{10} \\ 0 & \frac{4l}{15} & \frac{1}{10} & -\frac{l}{30} \\ -\frac{6}{5l} & \frac{1}{10} & \frac{6}{5l} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{l}{30} & \frac{1}{10} & \frac{2l}{15} \end{bmatrix}. \quad (1.321)$$

Following the same procedure as given in Example 1.12, we get

$$P_{cr} = (2.132) \frac{EI_0}{L^2}. \quad (1.322)$$

(ii) *Nonconsistent approach* Here, the geometric stiffness matrix is of the order (2×2) and the elastic stiffness matrix is of the order (4×4) . Further, the stiffness matrix given by (1.320) can be rewritten as

$$[K] = \frac{EI_0}{l^3} \begin{bmatrix} v_3 & v_5 & v_4 & v_6 \\ 18 & -\frac{15}{2} & l & -\frac{7}{2}l \\ -\frac{15}{2} & \frac{15}{2} & 4l & \frac{7}{2}l \\ l & 4l & 6l^2 & \frac{5}{4}l^2 \\ -\frac{7}{2}l & \frac{7}{2}l & \frac{5}{4}l^2 & \frac{9}{4}l^2 \end{bmatrix}. \quad (1.323)$$

Hence,

$$[K_{11}] = \frac{EI_0}{l^3} \begin{bmatrix} 18 & -\frac{15}{2} \\ -\frac{15}{2} & \frac{15}{2} \end{bmatrix},$$

$$[K_{12}] = [K_{21}]^T = \frac{EI_0}{l^3} \begin{bmatrix} l & -\frac{7}{2}l \\ 4l & \frac{7}{2}l \end{bmatrix},$$

$$[K_{22}] = \frac{EI_0}{l^3} \begin{bmatrix} 6l^2 & \frac{5}{2}l^2 \\ \frac{5}{2}l^2 & \frac{3}{2}l^2 \end{bmatrix}.$$

Substituting these in Eq. (1.312), we get

$$[K] = [K_{\text{eff}}] = \frac{EI_0}{191l^3} \begin{bmatrix} 2086 & -\frac{1221}{2} \\ -\frac{1221}{2} & \frac{481}{2} \end{bmatrix}. \quad (1.324)$$

Following the same procedure as discussed in Example 1.12, the geometric stiffness matrix we obtain is

$$[G] = \frac{P}{l} \begin{bmatrix} v_1 & v_3 & v_5 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \quad (1.325)$$

Hence,

$$P_{\text{cr}} = (2.174) \frac{EI_0}{l^2}. \quad (1.326)$$

If we divide the column into a large number of elements, we can make the solution converge to the exact value.

1.10 EFFECT OF SHEAR ON BUCKLING LOAD

In deriving the differential equation governing the buckling of a column, we assumed that the cross-sectional dimensions of the column are small as compared to its length, and therefore we neglected the effect of shear deformation on the buckling load. However, as the depth of a column increases, this effect has to be taken into account for a proper estimation of the critical load since shear deformation lowers the Euler (or critical) load. In what follows, we shall do this, keeping our treatment similar to that given in Timoshenko and Gere [3].

Consider a column subjected to an axial load P . At buckling, the column bends. Figure 1.24a shows the deformed configuration of an element of the column.

At any z along the column axis, the displacement along the Y -direction and the slope will be due to a combination of bending and shear.

The total slope is the algebraic sum of the bending slope and the shear slope (see Fig. 1.24b), i.e.,

$$\gamma = \psi - dv/dz. \quad (1.327)$$

The shear slope γ is given by

$$\gamma = \frac{nQ}{AG}, \quad (1.328)$$

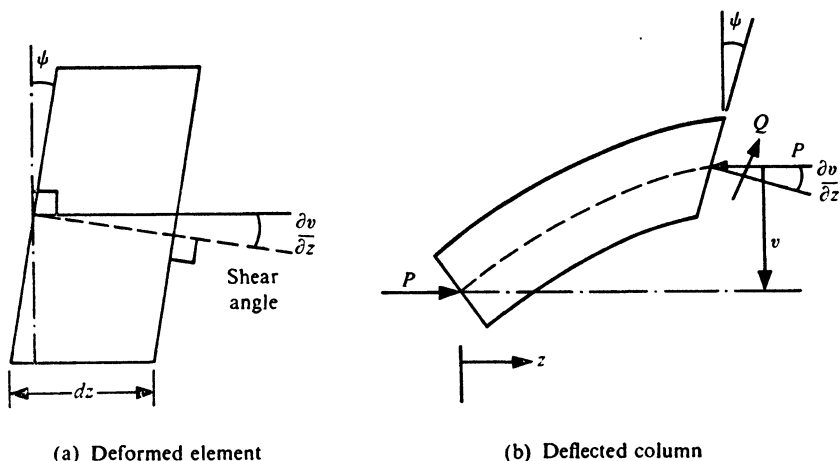


Fig. 1.24 Deformed Geometry of Column with Shear Effect.

where A is the area of cross-section and G is the shear modulus of the column material. The relation between the applied load P and the shear force Q is

$$Q = -P\psi.$$

Hence,

$$\gamma = -\frac{nP\psi}{AG}. \quad (1.329)$$

Substituting for γ from Eq. (1.329) in Eq. (1.327), we get

$$\frac{dv}{dz} = \psi \left[1 + \frac{nP}{AG} \right]. \quad (1.330)$$

From Eq. (1.58), the governing equation for a column we have is

$$\frac{d^2}{dz^2} (EI \frac{d\psi}{dz}) + P \frac{d^2 v}{dz^2} = 0.$$

Substituting for ψ and differentiating with respect to z , we get the governing equation for a uniform column as

$$\frac{EI}{1 + nP/(AG)} \frac{d^4 v}{dz^4} + P \frac{d^2 v}{dz^2} = 0. \quad (1.331)$$

Introducing the nondimensional parameter ξ , defined as $\xi = z/L$, we can write Eq. (1.331) as

$$\frac{d^4 v}{d\xi^4} + \lambda_n^2 \frac{d^2 v}{d\xi^2} = 0, \quad (1.332)$$

where

$$\lambda_n^2 = \frac{PL^2}{EI} \left(1 + \frac{nP}{AG}\right).$$

EXAMPLE 1.14 (Buckling load of simply-supported column with shear effect) From our earlier analysis [see Eq. (1.7)], for a simply-supported column, the first eigenvalue is π . Hence,

$$\frac{\pi^2}{L^2} = \frac{P_{cr}}{EI} \left(1 + \frac{nP_{cr}}{AG}\right). \quad (1.333)$$

Defining $P_E = \pi^2 EI / L^2$ as the Euler load without shear effect, we have P_{cr} with shear effect as

$$P_{cr} = \frac{\sqrt{1 + \frac{4nP_E}{AG}} - 1}{2n/(AG)}. \quad (1.334)$$

EXAMPLE 1.15 (Buckling load of simply-supported column with shear effect) Here, the strain energy of the column consists of bending and shear energies, given by

$$U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dz + \frac{1}{2} \int_0^L \frac{nQ^2}{2AG}, \quad (1.335)$$

$$V_e = -\frac{1}{2} \int_0^L P \left(\frac{dv}{dz}\right)^2 dz. \quad (1.336)$$

A one-term deflection curve satisfying the boundary conditions of the column can be taken as

$$v(z) = a_1 \sin \frac{\pi z}{L}. \quad (1.337)$$

We know

$$M = Pv, \quad Q = -P \frac{dv}{dz}. \quad (1.338)$$

Substituting Eqs. (1.337) and (1.338) in Eqs. (1.335) and (1.336) and applying Timoshenko's technique (see Section 1.8.1), we get

$$P_{cr} = P_E \frac{1}{1 + nP_E/(AG)}. \quad (1.339)$$

Thus, we see that the critical load here is less than that given by Eq. (1.32). This implies that, for purposes of design, we should take into account the effect of shear on the buckling load.

PROBLEMS

1.1 An aluminium pipe has inner and outer diameters of 0.03 m and 0.035 m, respectively.

If the pipe is one metre long with simply-supported ends and with $E = 68.64$ GPa, calculate (i) the critical load for elastic buckling and (ii) the average stress.

1.2 A 1-m-long aluminium bar has a rectangular cross-section 0.045 m wide and 0.01 m thick. The ends of the bar are simply-supported with respect to bending about the *stiff* axis but clamped with respect to bending about the *weak* axis. Determine the critical load for elastic buckling if $E = 68.64$ GPa.

1.3 Determine the critical load for a uniform column (see Fig. 1.25) clamped at one end and constrained at the other end, such that it is free to move along the Y -direction without rotation.

1.4 A steel bar of rectangular cross-section $0.02 \text{ m} \times 0.04 \text{ m}$ with hinged ends is compressed axially. Determine the minimum length of the bar if the stress at the limit of proportionality of steel is 196.2 MPa and $E = 206$ GPa. Also, compute the magnitude of the critical stress if the length of the bar is 1.5 m.

1.5 Solve Problem 1.4, assuming the bar to be of circular cross-section with 0.025 m diameter and with clamped ends.

1.6 Calculate the first critical load for a column with a spring of stiffness k attached to the column at the middle and simply-supported at two ends, as shown in Fig. 1.26. What will be the critical load for the second mode of buckling?

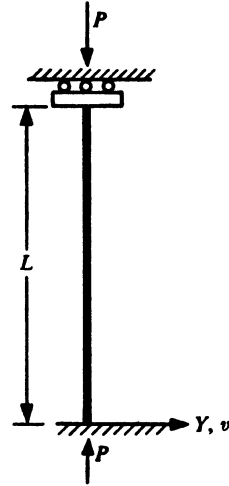


Fig. 1.25 Problem 1.3.

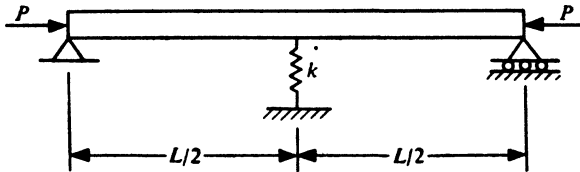


Fig. 1.26 Problem 1.6.

1.7 Find the buckling stress for a column, made of an aluminium alloy, with width = 0.02 m, depth = 0.0025 m, and length = 0.5 m. The column is simply-supported about the major bending axis and clamped about the minor bending axis.

1.8 Calculate the critical buckling load of a column of rectangular cross-section $0.02 \text{ m} \times 0.01 \text{ m}$ and supported as shown in Fig. 1.27. The supports B and C make a line contact with the cross-section. Given $E = 206$ GPa and column length = 1 m.

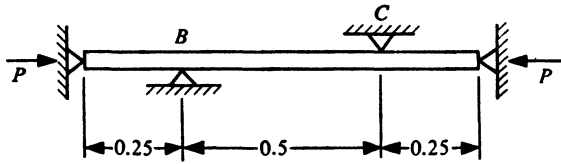


Fig. 1.27 Problem 1.8.

1.9 Obtain the differential equation and the boundary conditions for a column which is axially loaded and supported elastically as in Fig. 1.28. The elastic restraints are provided by rotational and extensional springs of stiffnesses k_a and k_v , respectively. (Hint: Use the variational principle.)

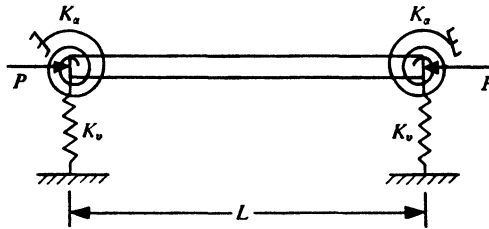


Fig. 1.28 Problem 1.9.

1.10 Solve the differential equation obtained in Problem 1.9, and get the expression for the critical load when:

- (i) k_a and k_v both tend to infinity.
- (ii) k_a tends to zero and k_v tends to infinity.

1.11 Obtain the expression for the buckling load of the column shown in Fig. 1.29, where

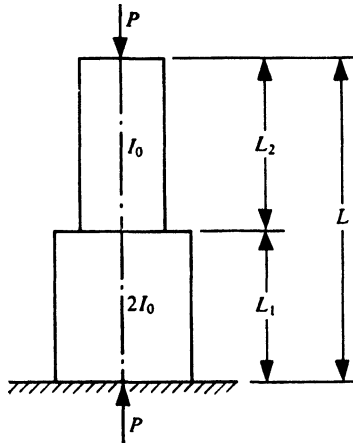


Fig. 1.29 Problem 1.11.

I_0 is the second moment of area of the column cross-section.

1.12 Obtain an expression for the buckling load of the column shown in Fig. 1.30. The column is attached to two elastic springs, each of stiffness k /unit deflection, and is hinged at one end and free at the other end.

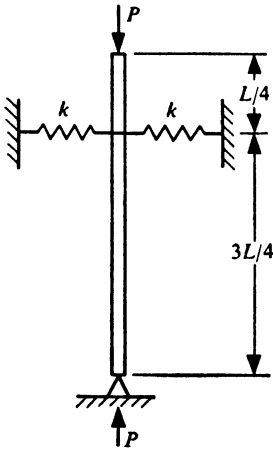


Fig. 1.30 Problem 1.12.

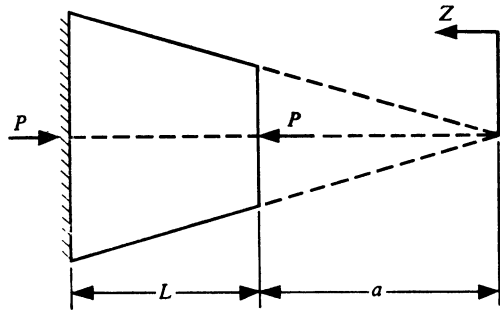


Fig. 1.31 Problem 1.13.

1.13 Obtain an expression for the critical load of a column (see Fig. 1.31) fixed at one end and free at the other end. The moment of inertia varies as $I_0(z/a)^4$.

1.14 Use the Rayleigh-Ritz method to obtain an approximate expression for the critical load of the stepped column shown in Fig. 1.32. The column is simply-supported. Check for the convergence of the solution.

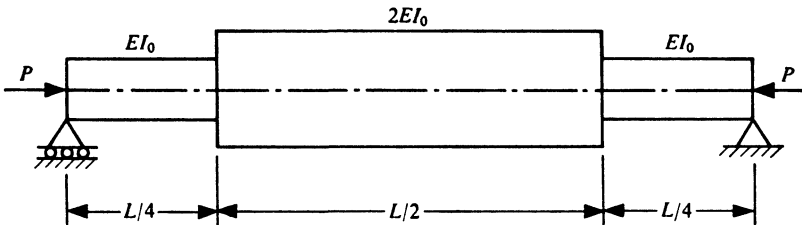


Fig. 1.32 Problem 1.14.

1.15 Solve Problem 1.14 by the finite difference technique.

1.16 Use the Rayleigh-Ritz method to determine the critical load of a fixed-fixed column (Fig. 1.33), supported along its entire length by an elastic foundation corresponding to a force k kg per unit length per unit deflection.

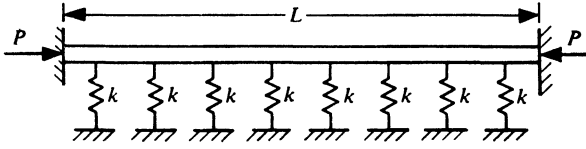


Fig. 1.33 Problem 1.16.

1.17 Use the Rayleigh-Ritz technique to obtain an approximate buckling load of a non-uniform simply-supported column (see Fig. 1.34) whose bending stiffness is given by $EI = EI_0[1 + (I_1/I_0) \sin(\pi z/L)]$.

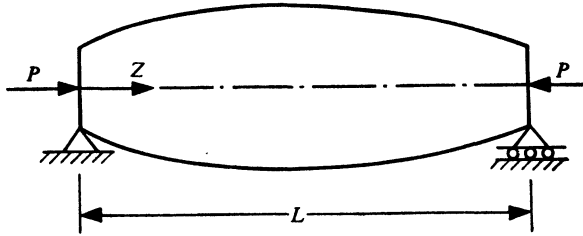


Fig. 1.34 Problem 1.17.

1.18 Solve Problem 1.17 by employing the Galerkin technique.

1.19 Use the finite element method to obtain the buckling load of a nonuniform column of circular cross-section (see Fig. 1.35) whose second moment of area varies as

$$I(z) = I_0(1 - \alpha \frac{z}{L})^4,$$

where α is the taper parameter defined as $(D - d)/D$.

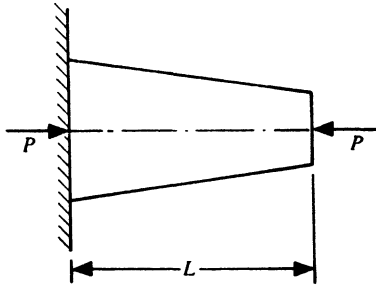


Fig. 1.35 Problem 1.19.

1.20 Use the finite element method to obtain the buckling load of a nonuniform column of rectangular cross-section (Fig. 1.36) with constant breadth and linear depth taper. The second moment of area varies as $I(z) = I_0[1 - \alpha(z/L)]^3$.

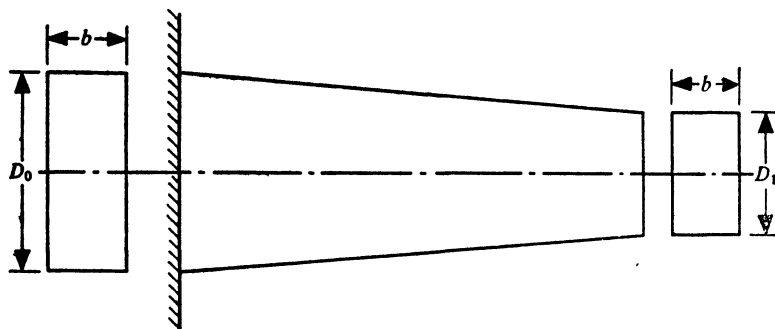


Fig. 1.36 Problem 1.20.

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2

Inelastic Buckling of Columns

2.1 INTRODUCTION

In Chapter 1, we considered the elastic buckling of columns and showed that columns undergo form failure with stresses within the elastic limit if the slenderness ratio is quite large, that is, approximately greater than 80 (see Fig. 2.1). When L_e/ρ is less than 20, a

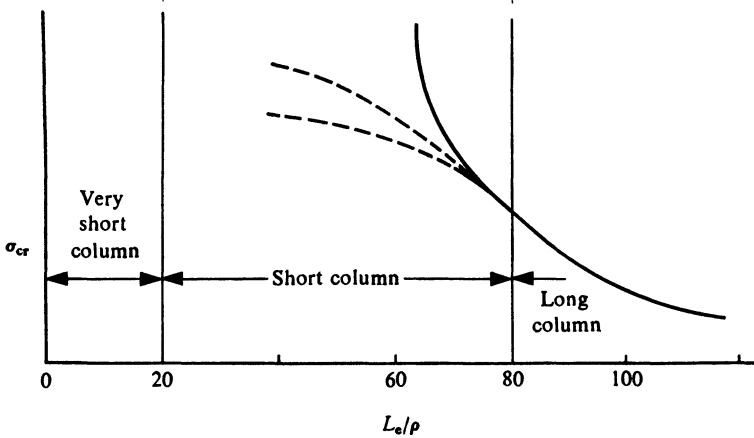


Fig. 2.1 Typical Column Curve for Thin-Walled Section.

column fails by crushing. This type of instability is observed in short struts, e.g., rivets. The region $20 < L_e/\rho < 80$ is termed as a short column; the stresses here exceed the elastic limit before form failure takes place. In such a situation, the Euler formula derived in Section 1.1 is not applicable and an alternative method of solution has to be used. If we examine the stress-strain curve for a metal (Fig. 2.2), we shall observe that the modulus has a unique value within the elastic limit. However, if the stress increases beyond this value, the modulus becomes a function of the stress. In other words, we should know the operating level of the stress before we can find out the modulus. This makes the analysis in the inelastic region complicated. For example, if the loading on a column is such that the stress lies in the region AB (Fig. 2.2), the modulus has a unique value corresponding to the tangent modulus. On the other hand, if the loading is such that the stress lies in the region BC , the modulus then depends on the location of the point P_2 on BC .

The first attempt to obtain the critical load of an inelastic column was made by F. Engesser who modified the Euler equation (see Bleich [1] and Gerard [2] for details), valid for an elastic column, by replacing E (elastic modulus) by E_t (tangent modulus). Around the same period, A. Considère (see Bleich [1]) modified the Euler equation by replacing E by E_R (reduced modulus) or E_D (double modulus) which lies between the elastic modulus and the tangent modulus. This implies that there exist two stages in the inelastic buckling of a column. The experiments von Karman later conducted (see Bleich [1]) on small specimens of rectangular cross-section columns of mild steel supported the observations made by Considère. Thus, the use of E_D for estimating the inelastic buckling load remained in vogue for a long time. Shanley [3], while experimenting with different shapes of

columns of an aluminium alloy, concluded that the results theoretically obtained by using E_D or E_R in the Euler formula are different from those arrived at experimentally. On the other hand, he found that the theoretical values obtained by applying E_t in the Euler formula are closer to the experimental values. (He attributed this to the differences in the stress-strain curve for steel and aluminium alloy.) He further observed that this is valid only in a situation where there is no strain reversal. In an instance where strain reversal takes place, the double modulus theory gives the correct result. This strain reversal occurs when axial straining and bending proceed simultaneously.

In what follows, we shall describe the (i) reduced (or double) modulus theory and (ii) tangent modulus theory.

2.2 DOUBLE MODULUS THEORY

For deriving the governing equation, we shall assume that the

- (i) deformations are small as compared to the cross-sectional dimensions (i.e., the strain-deformation relation is linear),
- (ii) plane sections after deformation remain plane and normal to the central line (that is, the shear deformation is neglected),
- (iii) stress-strain relation in any longitudinal fibre is given by the stress-strain diagram of the material, and
- (iv) cross-section is symmetric.

Consider a column with symmetric cross-section, as shown in Fig. 2.3. The column is subjected to an axial load P which gradually increases. Further, the column remains perfectly straight till the critical load P_{cr} is reached. At this load, the column bends slightly. The

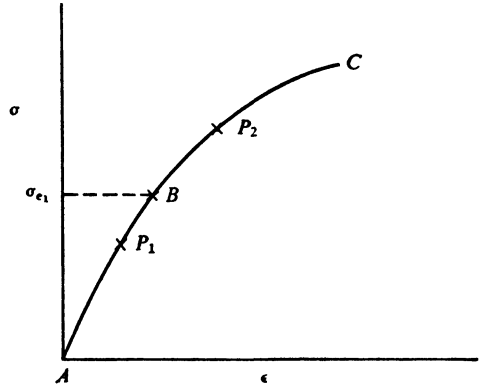


Fig. 2.2 Typical Stress-Strain Curve for Metal.

stress-strain diagram of the material is shown in Fig. 2.4. Also, the stress distribution at any cross-section is as depicted in Fig. 2.3.

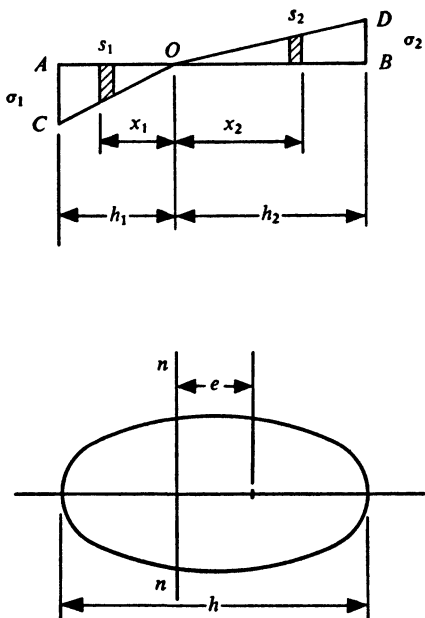


Fig. 2.3 Symmetrical Cross-Section and Stress Distribution.

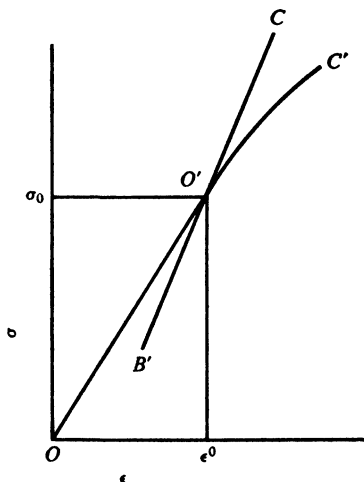


Fig. 2.4 Schematic Stress-Strain Diagram.

Assume that a stress σ_0 develops at every point in the cross-section before the column deflects. After deflection, the stress in the compression fibre increases, whereas that in the tension fibre effectively decreases though this stress still remains compressive. The loading takes place along the curved path $O'C'$ and the unloading takes place along the path $O'B'$, parallel to the initial slope of the stress-strain curve. Since it is difficult to represent the curved path $O'C'$ mathematically, we shall, in our analysis, take into account the slope of the line $O'C$ instead of that of $O'C'$. Thus, the loading is governed by the tangent modulus E_t , and the unloading by the elastic modulus E . In view of assumption (ii), the small bending stresses are superposed on the direct compressive stress and are distributed along the depth of the cross-section, as shown in Fig. 2.3.

The equilibrium equations for the bent configuration at any given section are

$$\int_0^{h_1} \sigma_a dA - \int_0^{h_2} \sigma_b dA = 0, \quad (2.1a)$$

$$\int_0^{h_1} \sigma_a (x_1 + e) dA + \int_0^{h_2} \sigma_b (x_2 - e) dA = Pv, \quad (2.1b)$$

where e is the distance between the centre of gravity and the shear centre. The deflection v

is measured with respect to the centroidal axis of the column.

From Fig. 2.3, we have

$$\sigma_a = \frac{\sigma_1}{h_1} x_1, \quad \sigma_b = \frac{\sigma_2}{h_2} x_2 \quad (2.2)$$

and, in view of assumption (i),

$$\sigma_1 = E h_1 \frac{d^2 v}{dz^2}, \quad \sigma_2 = E_t h_2 \frac{d^2 v}{dz^2}. \quad (2.3)$$

Substituting Eqs. (2.2) and (2.3) in Eq. (2.1a), we get

$$E \frac{d^2 v}{dz^2} \int_0^{h_1} x_1 dA - E_t \frac{d^2 v}{dz^2} \int_0^{h_2} x_2 dA = 0. \quad (2.4)$$

Equation (2.4) along with the relation $h_1 + h_2 = h$ determines the position of the neutral axis of the section. Equation (2.1b) yields

$$\frac{d^2 v}{dz^2} (E \int_0^{h_1} x_1^2 dA + E_t \int_0^{h_2} x_2^2 dA) = P v$$

or

$$(EI_1 + E_t I_2) \frac{d^2 v}{dz^2} = P v, \quad (2.5)$$

where I_1 and I_2 represent the second moment of the two portions of the area of the cross-section about the axis nn . In terms of the notation

$$E_D I = E_R I = EI_1 + E_t I_2,$$

we can write Eq. (2.5) as

$$E_R I \frac{d^2 v}{dz^2} = P v. \quad (2.6)$$

The value of E_R depends on the shape of the cross-section and the material properties. Equation (2.6) is of the same form as Eq. (1.2), and hence the method of solution here remains the same. Thus, for a simply-supported inelastic column, the first critical load P_r is given by

$$P_r = \pi^2 \bar{E} I / L^2, \quad (2.7)$$

where

$$\bar{E} = E \frac{I_1}{I} + E_t \frac{I_2}{I}.$$

2.3 TANGENT MODULUS THEORY

As already noted, the tangent modulus theory is based on the fact that there is no strain reversal when a column passes from the straight to a bent configuration. Shanley [3] showed

that the bending of a column may proceed simultaneously with increase in the axial load. According to him, there is a continuous spectrum of deflected configurations corresponding to the values of the axial load P between P_t (load based on tangent modulus theory) and P_r (load based on reduced modulus theory). The deflection v associated with a load has a definite value and increases from zero to infinity when P changes from P_t to P_r . Figure 2.5

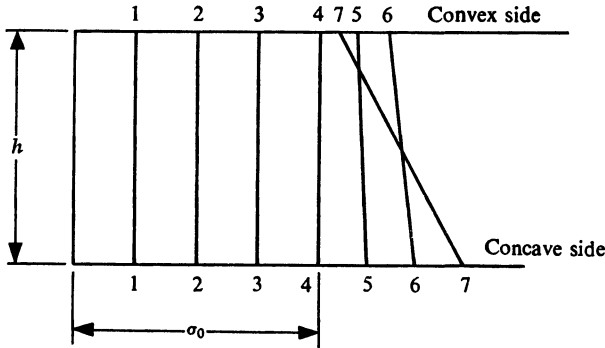


Fig. 2.5 Variation of Stress across Depth of Section.

shows the possible stress distribution across the depth of a section as P increases from zero to P_r . Here, during the load increments 1 to 4, the stress remains constant across the depth and corresponds to the straight configuration. Beyond 4, the column starts bending and the rate of increase of stress on the convex side decreases. This corresponds to loading at stages 5 and 6. If the load is further increased as in stage 7, the stress on the convex side decreases from the immediately preceding value, whereas on the concave side it increases. The modulus corresponding to stages 5 and 6 where there is no strain reversal is the local tangent modulus. Thus, Eq. (2.6) becomes

$$E_t I \frac{d^2 v}{dz^2} = P v. \quad (2.8)$$

In terms of the notation $\eta = E_t/E$, we may write Eq. (2.8) as

$$E I \eta \frac{d^2 v}{dz^2} = P v, \quad (2.9)$$

where η is the plasticity reduction factor. Hence, the lowest buckling load P_t for a pin ended uniform column in the inelastic range is

$$P_t = \pi^2 E I \eta / L^2. \quad (2.10)$$

For most metals, η lies between 0.8 and 0.95.

We shall now consider examples to illustrate the dependence of E_R on the cross-sectional shape.

EXAMPLE 2.1 (Reduced modulus of rectangular section) Consider a rectangular section

of depth h and width b . Figure 2.6 schematically shows the variation of stress and strain across the depth as also the stress-strain curve for the column material.

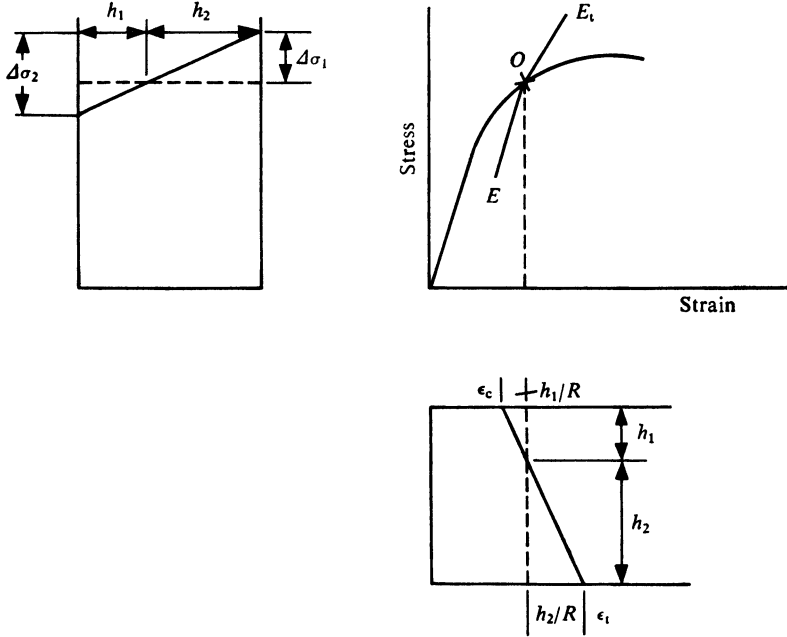


Fig. 2.6 Variation of Stress and Strain along Depth.

The equilibrium equations for the bent configuration at any given section are

$$\int_A \sigma dA = P, \quad (2.11)$$

$$\int_A \sigma x dA = Pv. \quad (2.12)$$

Assuming that the deflections are small, we find the strain ϵ is

$$\epsilon = \frac{x}{R} = -x \frac{d^2 v}{dz^2}, \quad (2.13)$$

where R is the radius of curvature. Equations (2.11) and (2.12) then reduce to

$$bR \int_{\epsilon_c}^{\epsilon_t} \sigma d\epsilon = P, \quad (2.14)$$

$$bR^2 \int_{\epsilon_c}^{\epsilon_t} \sigma \epsilon d\epsilon = Pv. \quad (2.15)$$

Although the stress-strain diagram has a curvature (see Fig. 2.6), we assume that the stress-strain curve at O is made up of two straight lines. Therefore, we have to satisfy two equilibrium equations, namely, (2.14) and (2.15). As the section is rectangular, these reduce to

$$\frac{1}{2}h_1\Delta\sigma_1 = \frac{1}{2}h_2\Delta\sigma_2, \quad (2.16)$$

$$\frac{1}{3}b\Delta\sigma_1h^2 + \frac{1}{3}b\Delta\sigma_2h_2^2 = Pv. \quad (2.17)$$

In addition, we have

$$h_1 + h_2 = h. \quad (2.18)$$

Since the stress-strain relation is linear, we get

$$\Delta\sigma_1 = E\Delta\epsilon_1 = E\frac{h_1}{R}, \quad (2.19a)$$

$$\Delta\sigma_2 = E_t\Delta\epsilon_2 = E_t\frac{h_2}{R}. \quad (2.19b)$$

Substituting Eqs. (2.19) in Eq. (2.16), we obtain

$$h_1^2E = h_2^2E_t. \quad (2.20)$$

Equations (2.20) and (2.18) lead to

$$h_1 = \frac{h\sqrt{E_t}}{\sqrt{E} + \sqrt{E_t}}, \quad (2.21a)$$

$$h_2 = \frac{h\sqrt{E}}{\sqrt{E} + \sqrt{E_t}}. \quad (2.21b)$$

Equations (2.21) show that the position of the neutral axis depends on the column material property. Substituting Eqs. (2.19) in Eq. (2.17), we have

$$\frac{1}{3}\frac{Eh_1^3b}{R} + \frac{1}{3}\frac{E_th_2^3b}{R} = Pv. \quad (2.22)$$

Substituting in Eq. (2.22) for h_1 and h_2 in terms of h , we get

$$Pv = \frac{1}{12} \frac{bh^3}{R} \frac{4EE_t}{(\sqrt{E} + \sqrt{E_t})^2}. \quad (2.23)$$

Comparing this with Eq. (2.6), we find that

$$E_R = \frac{4EE_t}{(\sqrt{E} + \sqrt{E_t})^2}. \quad (2.24)$$

If $E = E_t$, which is true in the elastic region, Eq. (2.24) reduces to $E_R = E$.

EXAMPLE 2.2 (Reduced modulus of I -section with negligible web thickness) Consider an I -section in which one half the cross-sectional area is lumped at each flange (see Fig. 2.7)

and the area of the web is neglected. (For another discussion on this problem, see Timoshenko and Gere [4] and Chajes [5].)

The equilibrium equations here are

$$\Delta\sigma_{12} = \Delta\sigma_{22}, \quad (2.25a)$$

$$\Delta\sigma_{12}h_1 + \Delta\sigma_{22}h_2 = Pv. \quad (2.25b)$$

Substituting for $\Delta\sigma_1$ and $\Delta\sigma_2$ from Eqs. (2.19) in Eqs. (2.25), we get

$$\frac{E}{E_t} = \frac{h_2}{h_1}, \quad (2.26a)$$

$$\frac{A}{2R}(Eh_1^2 + E_th_2^2) = Pv. \quad (2.26b)$$

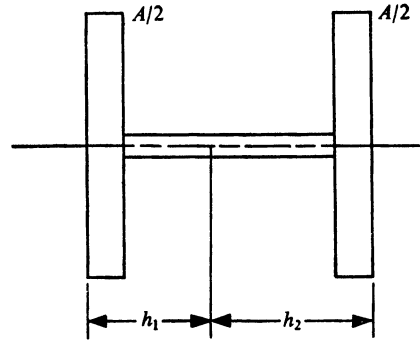


Fig. 2.7 Idealized I-Section.

Using Eq. (2.18) in Eqs. (2.26), we obtain

$$h_1 = \frac{hE_t}{E + E_t}, \quad (2.27a)$$

$$h_2 = \frac{hE}{E + E_t}, \quad (2.27b)$$

$$\frac{Ah^2}{2R} \frac{EE_t}{E + E_t} = Pv. \quad (2.27c)$$

Equation (2.27c) can be rewritten as

$$Pv = \frac{I}{R} \frac{2EE_t}{E + E_t}, \quad (2.28)$$

where $I = A(h/2)^2$. Thus, the reduced modulus E_R is given by

$$E_R = \frac{2EE_t}{E + E_t}. \quad (2.29)$$

2.4 ECCENTRICALLY LOADED COLUMNS

Earlier in this chapter, we assumed that the axial load passes through the centroid of the column cross-section. In practice, there are instances where this cannot be realized. Here, we shall therefore consider such situations, i.e., the eccentrically loaded columns. In doing so, we shall assume that the stress levels in the sections exceed the proportional limit.

In an eccentrically loaded column, the stress-strain relation varies in a complex manner from point to point. As such, it is not possible to get a closed-form solution. Therefore, an approximate numerical solution has to be obtained. One such technique has been suggested by von Karman (see Bleich [1]). Since this technique is quite involved, we shall confine our

discussion to the one proposed by Chajes [5]. It is based on the following assumptions:

- (i) The column deflects as a sine wave.
- (ii) The stress varies linearly across the section. (The actual stress-strain curve of the column material is used to determine the maximum fibre stresses before making this assumption.)
- (iii) After deformation, the plane sections remain plane and normal to the central line.
- (iv) The deformations are small (i.e., the moment-curvature relation is linear).

Consider the eccentrically loaded column shown in Fig. 2.8. It is initially straight and is of rectangular cross-section of width b and depth h . The column cross-section is subjected to both axial and bending stresses which increase as P increases. Figure 2.9 shows the stress

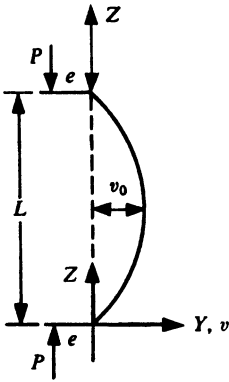


Fig. 2.8 Eccentrically Loaded Column.

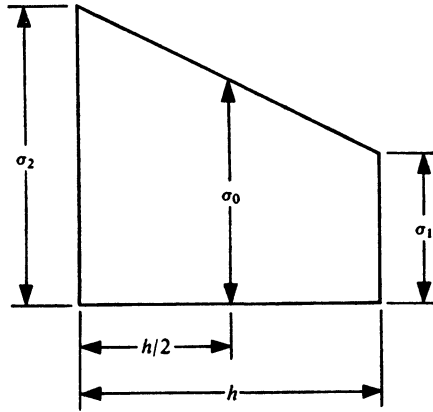


Fig. 2.9 Variation of Stress across Any Section.

variation across any section. Here, σ_0 is the stress due to the axial load and σ_1, σ_2 are the extreme fibre stresses including bending. If ϵ_1 and ϵ_2 are the strains corresponding to the stresses σ_1 and σ_2 , respectively, then

$$\frac{1}{R} = \frac{\epsilon_1 - \epsilon_2}{h}, \quad (2.30)$$

where R is the radius of curvature. In view of assumption (iv), we have

$$\frac{d^2v}{dz^2} = \frac{\epsilon_1 - \epsilon_2}{h}. \quad (2.31)$$

Defining the modulus $\bar{E}$ as

$$\bar{E} = \frac{\sigma_2 - \sigma_1}{\epsilon_2 - \epsilon_1}, \quad (2.32)$$

we can write Eq. (2.31) as

$$\frac{d^2v}{dz^2} = \frac{\sigma_1 - \sigma_2}{\bar{E}h}. \quad (2.33)$$

Now,

$$\sigma_1 = \frac{P}{A} - \frac{Mh}{2I}, \quad \sigma_2 = \frac{P}{A} + \frac{Mh}{2I}, \quad (2.34)$$

where A is the area of the cross-section, I the second moment of area from the centroidal axis, and M the bending moment at any section. From Eq. (1.1), we have

$$\frac{d^2v}{dz^2} = -\frac{M}{\bar{E}I}. \quad (2.35)$$

The moment M is equal to $P(e + v)$, the deflection v being measured from the straight configuration. Following assumption (i), we see that the deformed shape of the column is given by

$$v = v_0 \sin \frac{\pi z}{L}. \quad (2.36)$$

Substituting relation (2.36) in Eq. (2.35) and evaluating v_0 at $z = L/2$, we get

$$v_0 = \frac{L^2}{\pi} \frac{P(e + v_0)}{\bar{E}I} \quad (2.37)$$

and, because $I = Ak^2$,

$$v_0 = \frac{e}{\left(\frac{\pi^2 \bar{E}}{(L/k)^2 \sigma_0} - 1 \right)}. \quad (2.38)$$

In Eq. (2.38), $\bar{E}$ is still an unknown.

The maximum and minimum stresses at $z = L/2$ are given by

$$\sigma_2 = \frac{P}{A} + \frac{P(e + v_0)h}{2I}, \quad (2.39a)$$

$$\sigma_1 = \frac{P}{A} - \frac{P(e + v_0)h}{2I}. \quad (2.39b)$$

Substituting for v_0 from Eq. (2.38) in expressions (2.39), we obtain

$$\sigma_2 = \sigma_0 \left[1 + \frac{eh}{k^2} \left\{ \frac{1}{1 - \frac{\sigma_0(L/k)^2}{\pi^2 \bar{E}}} \right\} \right], \quad (2.40a)$$

$$\sigma_1 = \sigma_0 \left[1 - \frac{2h}{k^2} \left\{ \frac{1}{1 - \frac{\sigma_0(L/k)^2}{\pi^2 \bar{E}}} \right\} \right]. \quad (2.40b)$$

From Eqs. (2.40), we can obtain $\bar{E}$ provided we know σ_2 which itself is an unknown. To evaluate σ_2 , an iterative technique is used. That is, $\bar{E}$ is assumed to have some value, and subsequently σ_1 , σ_2 , and hence ϵ_1 , ϵ_2 , are estimated from the stress-strain curve. The new value of $\bar{E}$ is then gotten, using Eq. (2.32).

Once we know $\bar{E}$, we can obtain v_0 from Eq. (2.37). We can then plot P versus v_0 by considering different values of P .

We shall demonstrate the application of the technique we have discussed through an example.

EXAMPLE 2.3 (Deflection of I -section column in inelastic range) Consider an idealized I -section column of an aluminium alloy as shown in Fig. 2.10, and assume that the effect of the web is negligible. Further, the I -section is symmetric with a flange area 0.013 m^2 . Also, the flanges are separated by a distance 0.10 m . The slenderness ratio (L/k) of the column is taken to be 30 and the eccentricity e to be 0.025 m , and σ_0 (for the column material) $= 1.274 \times 10^9 \text{ N/m}^2$.

The stress σ_2 from Eq. (2.40a) is

$$\sigma_2 = \sigma_0 \left[1 + \frac{eh}{k^2} \left\{ \frac{1}{1 - \frac{\sigma_0(2/k)^2}{\pi^2 \bar{E}}} \right\} \right].$$

Substituting the given values of e , h , k , ... in this equation, we get

$$\sigma_2 = 1.274 \times 10^9 \left[1 + 0.2 \frac{1}{1 - \frac{1.274 \times 10^9 \times 900}{\pi^2 \bar{E}}} \right] \text{ N/m}^2.$$

Assume that $\sigma_2 = 1.883 \times 10^9 \text{ N/m}^2$. Then,

$$\begin{aligned} \sigma_1 &= 1.274 - (1.883 - 1.274) \times 10^9 \\ &= 0.667 \times 10^9 \text{ N/m}^2. \end{aligned}$$

For these values of σ_2 and σ_1 , the corresponding strains from the stress-strain curve (Fig. 2.11) are

$$\epsilon_1 = 0.0012, \quad \epsilon_2 = 0.003.$$

Hence, from Eq. (2.32),

$$\bar{E} = 0.675 \times 10^{12} \text{ N/m}^2.$$

Substituting in Eq. (2.40a) this value of $\bar{E}$, we get the new value of σ_2 as

$$\sigma_2 = 1.547 \times 10^9 \text{ N/m}^2.$$

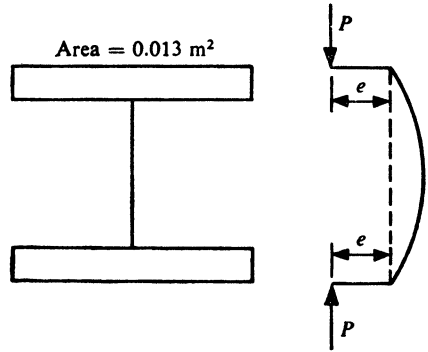


Fig. 2.10 Eccentrically Loaded I -Section Column.

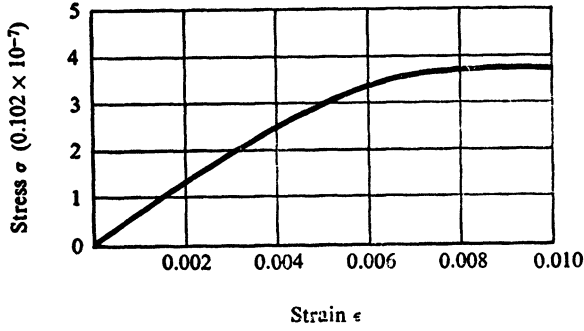


Fig. 2.11 Typical Stress-Strain Curve for Aluminium Alloy.

As a second approximation, take $\sigma_2 = 1.5 \times 10^9 \text{ N/m}^2$. Then,

$$\sigma_1 = 0.961 \times 10^9 \text{ N/m}^2.$$

From the stress-strain curve (Fig. 2.11), the strains corresponding to the new values of σ_1 and σ_2 are

$$\epsilon_1 = 0.0016, \quad \epsilon_2 = 0.0027.$$

Hence,

$$\bar{E} = \frac{(1.547 - 0.961) \times 10^9}{0.0027 - 0.0016} = 0.532 \times 10^{12} \text{ N/m}^2.$$

Substituting in Eq. (2.40a) this value of $\bar{E}$, we get

$$\sigma_2 = 1.595 \times 10^9 \text{ N/m}^2.$$

As a third approximation, take $\sigma_2 = 1.595 \times 10^9 \text{ N/m}^2$. Therefore,

$$\sigma_1 = 0.954 \times 10^9 \text{ N/m}^2.$$

As can be observed, the solution has converged to the second decimal place. We can either stop at this stage or proceed as before, depending on the accuracy we desire. Now, the deflection at the centre can be obtained, using Eq. (2.38), as

$$v_0 = \frac{e}{\frac{\pi^2 \bar{E}}{(L/k)^2} - 1} = \frac{0.025}{\frac{\pi^2 \times 0.532 \times 10^{12}}{1.274 \times 10^9 \times 900} - 1} = 0.006 \text{ m.}$$

2.5 EMPIRICAL RELATIONS FOR SHORT COLUMNS

In Sections 2.2 and 2.3, we noted that for estimating the buckling load in the inelastic region a knowledge of the stress-strain curve of the column material is required. This makes the task difficult since the stress-strain curve for a given material may not be readily available and a considerable effort may have to be put in to obtain it. Further, it is not easy to express

the relation between the allowable column stress σ_c and the slenderness ratio (L'/ρ) , where L' is defined as the effective length of the column ($L' = L/\sqrt{c}$, c being called the fixity coefficient of the column), in a simple manner. For this, a large number of experiments have been conducted and a number of simple approximate equations for a wide variety of materials are now available. Such approximate equations are reasonably close to the results obtained by column tests. Of these, the most commonly used empirical relationship is a power law of the form

$$\sigma_c = \sigma_{co} - K_e(L'/\rho)^n, \quad (2.41)$$

where σ_{co} is the stress intercept at $(L'/\rho) = 0$ and n is a parameter. Both these quantities depend on the material and also on its condition (heat treated, cold worked, ...). The coefficient K_e is found by requiring the empirical curve and the Euler curve (elastic buckling) to be tangent at the transitional slenderness ratio $(L'/\rho)_{tr}$. Further, the value of the column stress σ_c , as obtained from the empirical curve and the Euler curve, should be the same at $(L'/\rho)_{tr}$.

From Eq. (1.12), the column stress predicted by the Euler curve is

$$\sigma_c = \frac{\pi^2 E}{(L'/\rho)^2}. \quad (2.42)$$

Equating the two stresses at the point $(L'/\rho)_{tr}$, we get

$$\sigma_{co} - K_e(L'/\rho)_{tr}^n = \frac{\pi^2 E}{(L'/\rho)_{tr}^2} \quad (2.43)$$

and, for equal slopes at this point, we obtain

$$-K_e n (L'/\rho)_{tr}^{n-1} = -\frac{2\pi^2 E}{(L'/\rho)_{tr}^3}. \quad (2.44)$$

Simplifying Eq. (2.44), we have

$$K_e = \frac{2\pi^2 E}{n(L'/\rho)_{tr}^{n+2}}. \quad (2.45)$$

Hence,

$$\sigma_{co} = \frac{\pi^2 E}{(L'/\rho)_{tr}^2} \left(1 + \frac{2}{n}\right). \quad (2.46)$$

We can rewrite Eq. (2.46) as

$$(L'/\rho)_{tr} = \pi \left[\frac{E}{\sigma_{co}} \left(1 + \frac{2}{n}\right) \right]^{1/2}. \quad (2.47)$$

Substituting this value of $(L'/\rho)_{tr}$ in Eq. (2.45), we obtain

$$K_e = \frac{2E\pi^2}{n\pi^{n+2} \left[\frac{E}{\sigma_{co}} \left(1 + \frac{2}{n}\right) \right]^{(n+2)/2}}. \quad (2.48)$$

Hence,

$$\sigma = \sigma_{co} - \frac{2E}{n\pi^n \left[\frac{E}{\sigma_{co}} \left(1 + \frac{2}{n} \right) \right]^{(n+2)/2}} \left(\frac{L'}{\rho} \right)^n. \quad (2.49)$$

By giving varying values to n in Eq. (2.49), we can obtain different empirical relations. Of these, the ones that are largely used, as given in Peery [6], are:

(i) Straight line formula ($n = 1$)

$$K_e = \frac{2}{\pi(E/\sigma_{co})^{1/2}} \frac{\sigma_{co}}{3^{3/2}}, \quad (2.50a)$$

$$\sigma_c = \sigma_{co} \left[1 - \frac{0.385}{\pi(E/\sigma_{co})^{1/2}} (L'/\rho) \right]. \quad (2.50b)$$

This fits in for most columns made of aluminium alloys.

(ii) Johnson's parabolic formula ($n = 2$)

$$K_e = \frac{E}{4\pi^2(E/\sigma_{co})^2}, \quad (2.51a)$$

$$\sigma_c = \sigma_{co} \left[1 - \frac{0.25}{\pi^2(E/\sigma_{co})} (L'/\rho)^2 \right]. \quad (2.51b)$$

Heat treated alloy steels with high values of ultimate tensile strength can be represented by these relations.

(iii) Semi-cubic formula ($n = 1.5$)

$$K_e = \frac{0.3027E}{\pi^{1.5}(E/\sigma_{co})^{1.75}}, \quad (2.52a)$$

$$\sigma_c = \sigma_{co} \left[1 - 0.3027 \left(\frac{L'/\rho}{\pi\sqrt{E/\sigma_{co}}} \right)^{1.5} \right]. \quad (2.52b)$$

Heat treated alloy steels with low values of ultimate tensile strength can be represented by these relations.

The Euler curve and each of these relations can be expressed in an elegant way. By defining the nondimensional parameters

$$B = \frac{L'/\rho}{\pi\sqrt{E/\sigma_{co}}}, \quad R_a = \frac{\sigma_c}{\sigma_{co}}, \quad (2.53)$$

we can write Eqs. (2.50b), (2.51b), (2.52b) as

$$R_a = 1.0 - 0.385B, \quad (2.54a)$$

$$R_a = 1.0 - 0.25B^2, \quad (2.54b)$$

$$R_a = 1.0 - 0.3027B^{1.5} \quad (2.54c)$$

and the Euler curve as

$$R_a = 1/B^2. \quad (2.54d)$$

Figure 2.12 schematically shows the variation of R_a with B .

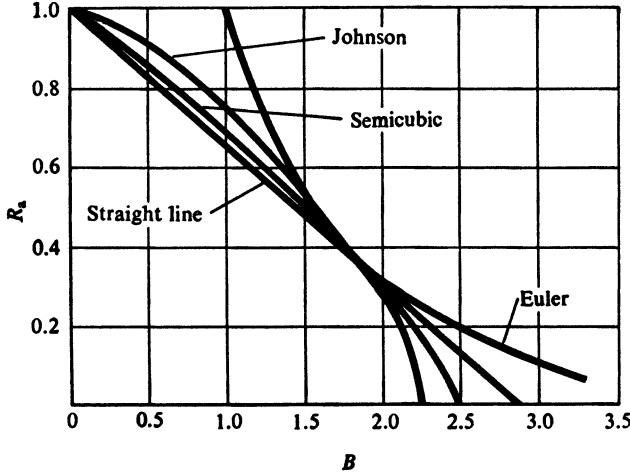


Fig. 2.12 Variation of R_a with B .

EXAMPLE 2.4 (Critical stress σ_c for aluminium alloy) Consider a column of hollow circular cross-section $0.025 \text{ m} \times 0.001 \text{ m}$ and length 1 m made out of an aluminium alloy. It is so supported that its fixity coefficient $c = 4$. Let us find its ultimate strength when $E = 68.68 \times 10^9 \text{ N/m}^2$ and the inelastic column curve $\sigma_c = 0.236 \times 10^9 - K_c(L'/\rho)$.

The effective length of the column is

$$L' = \frac{L}{\sqrt{e}} = \frac{L}{\sqrt{4}} = \frac{1}{2} = 0.5 \text{ m}.$$

The radius of gyration is

$$\rho = \sqrt{\frac{I}{A}} = 0.0086.$$

The slenderness ratio is

$$L/k = \frac{0.5}{0.0086} = 57.636.$$

From Eq. (2.50b), we obtain

$$\sigma_c = \sigma_{co} \left[1 - \frac{0.385}{\pi(E/\sigma_{co})^{1/2}} (L'/\rho) \right]$$

$$\begin{aligned}
 &= 0.2326 \times 10^9 \left[1 - \frac{0.385}{\pi \left(\frac{68.68}{0.2326} \right)^{1/2}} (57.636) \right] \\
 &= 0.2326 \times 10^9 (1 - 0.411) \\
 &= 0.2326 \times 10^9 \times 0.589 \\
 &= 0.137 \times 10^9 \text{ N/m}^2.
 \end{aligned}$$

PROBLEMS

2.1 Derive the expressions for the reduced modulus E_R for the sections shown in Figs. 2.13a and 2.13b.

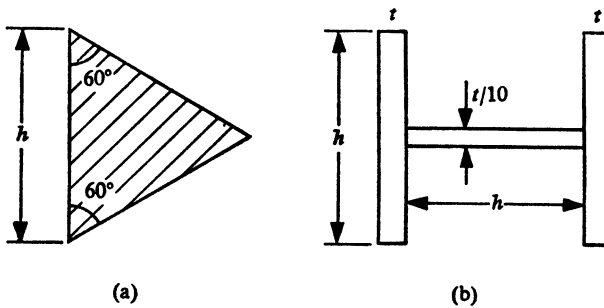


Fig. 2.13 Problem 2.1.

2.2 Obtain the reduced modulus E_R for the section in Fig. 2.13b, assuming that the web area is lumped at the flanges and that the flanges are connected by a thin member.

2.3 The short column curve for a material is represented by the equation

$$\sigma_c = \sigma_{co} - K_e (L'/\rho)^{1.8}.$$

Determine the value of K_e in order that this curve is tangent to the Euler curve

$$\sigma_c = \pi^2 E / (L'/\rho)^2.$$

Further, calculate the value of critical (L'/ρ) and the value of σ_c at this critical (L'/ρ) .

2.4 Find the column strength of a $0.0375 \text{ m} \times 0.0015 \text{ m}$ round alloy steel tube with length 0.75 m . The stress σ_{co} for this steel at $(L'/\rho) = 0$ is 1.037 GPa . Assume the end fixity coefficient to be 2.

2.5 Find the ultimate column strength of a round 24-ST aluminium tube $0.025 \text{ m} \times 0.00125 \text{ m}$ with length equal to 0.5 m and end fixity coefficient 1.5. The stress σ_{co} is 440.5 MPa .

2.6 A forging of an aluminium alloy has a cross-section shown in Fig. 2.14. Find the reduced modulus E_R and hence the buckling load. All dimensions are in metres.

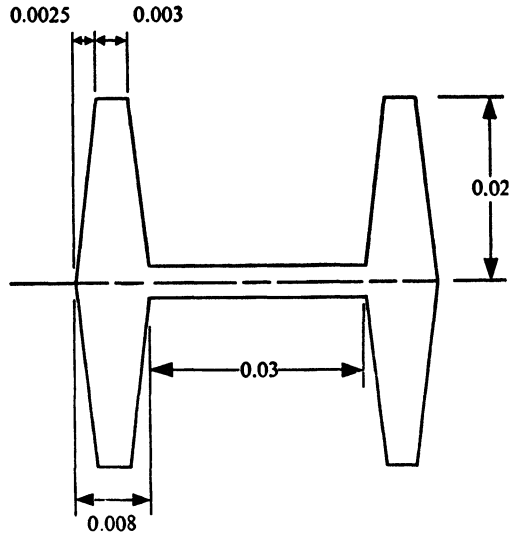


Fig. 2.14 Problem 2.6.

2.7 Obtain the buckling load for the section in Fig. 2.14 by using the straight line formula with $\sigma_{co} = 393.4$ MPa and $E = 66.72$ MPa.

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- [1] Bleich, F., *Buckling Strength of Metal Structures*, McGraw-Hill, New York, 1952.
- [2] Gerard, G., *Introduction to Structural Stability Theory*, McGraw-Hill, New York, 1962.
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3

Buckling of Open Sections

3.1 INTRODUCTION

In Chapters 1 and 2, we assumed that, for a column, the bending rigidity EI is much smaller than the torsional rigidity GJ . If this is so, there is a possibility of failure under flexure. Closed thin-walled and solid sections are prone to this type of failure. Further, the columns of these sections buckle in the plane of principal axes and do not undergo rotation. Such behaviour can be reasonably assumed for a section having two axes of symmetry, but this is not true when only one of the principal axes is an axis of symmetry or when there is no symmetry. When an open section, e.g., a channel section, an H -section, or an angle section, is subjected to an axial compressive load, it simultaneously bends and twists. This type of form failure occurs because of low torsional rigidity in such sections. Further, in any of these sections, the critical load for the buckled mode lies between the critical load for the pure torsional mode and that for the pure flexural mode. A pure torsional mode exists in a section whose centroidal axis coincides with the shear centre axis. For any general section, a combined torsion-flexure mode occurs.

In this chapter, we shall consider torsional buckling and torsional-flexural buckling of axially loaded columns.

3.2 TORSIONAL LOAD-DEFORMATION CHARACTERISTICS

It is well-known that when a torque is applied to a structural member whose cross-section is noncircular, the cross-section twists and warps. If warping is freely allowed, then the applied torque is resisted completely by the St. Venant shearing stresses. On the other hand, if the member is restrained from warping, then the torque is resisted partly by the St. Venant shearing stresses and partly by the stresses produced due to the constraint on warping. As we shall show later in this section, the relative magnitudes of the two stresses depend on the geometry of the cross-section and the ratio of the elastic modulus to the shear modulus, i.e., E/G .

The total torque T is written as

$$T = T_J + T_T, \quad (3.1)$$

where T_J is the St. Venant torque and T_T is the torsion-bending torque or the torque resisted by the axial stresses generated as a result of the constraint on warping.

The distribution of the torque T for an I -section, constrained against warping at the fixed

end and free at the other end, is schematically shown in Fig. 3.1.

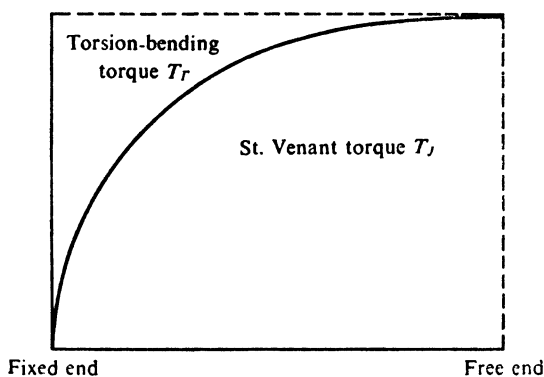


Fig. 3.1 Variation of Torque with Axis.

If a bar of constant circular section is subjected to a torque T , the torque is completely resisted by the circumferential shear stresses whose magnitude varies as their distance from the centre of the section. This implies that the shear stresses are maximum at the periphery and zero at the centre of the section. It is for this reason that a hollow shaft is preferred to a solid shaft for a given weight. For such a shaft, the angle of twist β is related to the torque T (see Peery [1]) as

$$T = GJ \frac{d\beta}{dz}, \quad (3.2)$$

where J is the St. Venant torsional constant and is equal to the polar moment of inertia I_p for a circular shaft, G the shear modulus of rigidity, and z the axial coordinate. For a non-circular closed section as well as an open section, J is obtained by using the membrane analogy (Williams [2]). For an open section shaft in the form of an elongated rectangle where the length is several times the thickness, the shear stresses are parallel to the edges of the cross-section and are maximum at the edges. The torsional constant J for such a section is given by

$$J = \frac{1}{3} \sum_{i=1}^n s_i t_i^3, \quad (3.3)$$

where s_i is the mean length of the i -th element, t_i the thickness of the i -th element, and n the number of elements in which the structure can be approximated by elongated rectangles. The expressions for J for some typical cross-sections are given in Table 3.1.

The membrane analogy can also be used for estimating J for a thin-walled closed section. For such a section, J is given by

$$J = \frac{4A^2}{\oint \frac{ds}{t}}, \quad (3.4)$$

Table 3.1 Values of J and I for Some Typical Sections

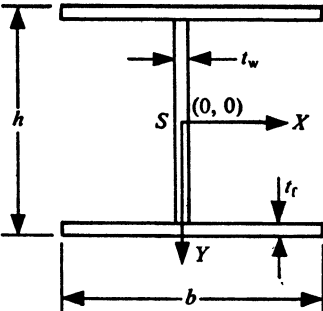
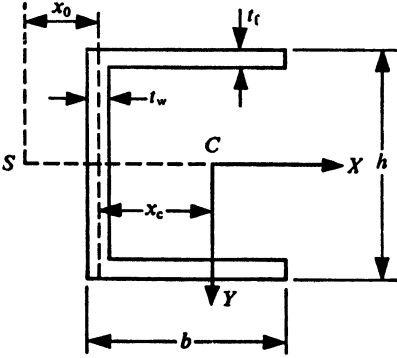
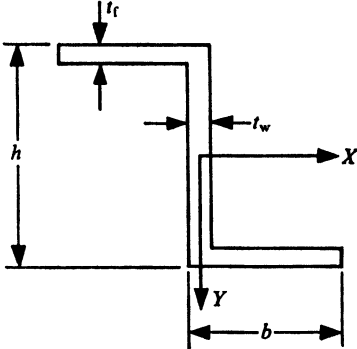
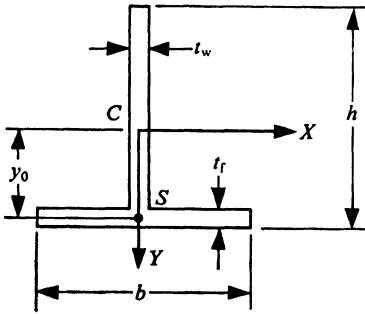
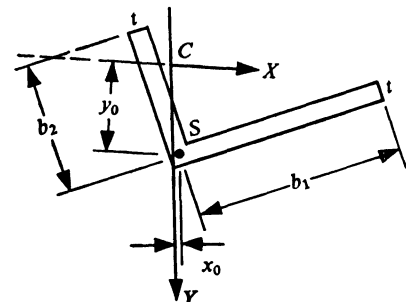
Cross-section	Shear centre	J ($2J_1 + J_2$)	I
I-section			
	(0, 0)	$J_1 = \frac{1}{3} b t_f^3,$ $J_2 = \frac{1}{3} h t_w^3$	$\frac{1}{24} t_f h^2 b^3$ (about S)
Channel section			
	$(x_c + x_0, \frac{h}{2}),$ $x_0 = b \frac{3b/h}{1 + 6b/h}$	$J_1 = \frac{1}{3} b t_f^3,$ $J_2 = \frac{1}{3} h t_w^3$	$\frac{t_f h^2 b^3}{12} \frac{2 + 3b/h}{1 + 6b/h}$ (about S)
Z-section			
	(0, 0)	$J_1 = \frac{1}{3} b t_f^3,$ $J_2 = \frac{1}{3} h t_w^3$	$\frac{t_f b^3 h^2}{12} \frac{2 + b/h}{1 + 2b/h}$ (about S)

Table 3.1 Values of J and Γ for Some Typical Sections (cont.)

Cross-section	Shear centre	J ($2J_1 + J_2$)	Γ
Inverted T-section			
	$(0, y_0)$	$J_1 = \frac{b}{2}t_f^3,$ $J_2 = \frac{1}{3}ht_w^3$	$\frac{t_f^3h^3}{36}(1 + \frac{(b/h)^3}{4})$ (about S)
Unequal angle			
	(x_0, y_0)	$J_1 = \frac{1}{3}b_2t^3,$ $J_2 = \frac{1}{3}b_1t^3$	$\frac{t^3}{36}(b_1^3 + b_2^3)$

where A is the area enclosed by the mean line, s the peripheral length, and t the thickness. For the steps needed in developing Eq. (3.4), see Williams [2].

For a bar of a noncircular section, if warping is freely allowed, no axial stresses develop. However, for a constrained section, the axial stresses in the flanges vary from zero at the free end to a maximum at the fixed end (see Fig. 3.2). The constraint on warping deforma-

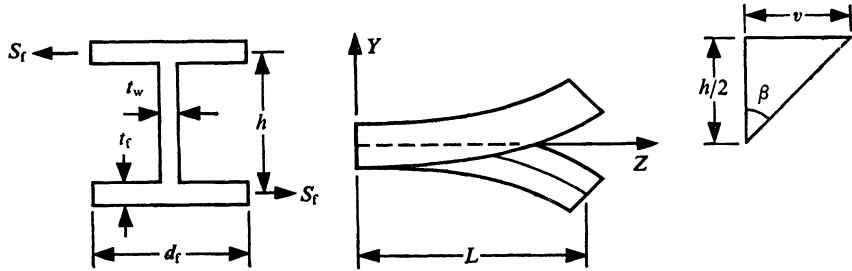


Fig. 3.2 Twisting of I-Section Restrained from Warping.

tion results in a differential bending of the flanges, one flange bending to the right and the other to the left. The shear forces S_f in the flange members form a couple. This couple is referred to as the torsion-bending torque, T_r .

To derive an expression for T_r , let us consider a beam of I -section of length L , subjected to a torque T at the free end (see Fig. 3.2). Since the section is not allowed to warp, the applied torque T is resisted partly by the St. Venant torque T_J and partly by the torsion-bending torque T_r . If the top flange is an independent cantilever beam of width t_f , depth d_f , subjected to a shear load S_f , then the torsion-bending torque T_r is

$$T_r = S_f h, \quad (3.5)$$

where

$$S_f = dM_f/dz. \quad (3.6)$$

Defining v as the lateral displacement of the flange, we find the moment M_f is given by

$$M_f = -EI_f \frac{d^2 v}{dz^2}, \quad (3.7)$$

where I_f is the second moment of area of the flange about the Y -axis. Assuming the deformation to be small, we see that v is related to β as

$$v = \frac{h}{2}\beta. \quad (3.8)$$

Therefore,

$$M_f = -EI_f \frac{h}{2} \frac{d^2 \beta}{dz^2}. \quad (3.9)$$

Substituting the expression for M_f in Eq. (3.6) and then substituting the result so obtained in Eq. (3.5), we get

$$T_r = -EI_f \frac{h^2}{2} \frac{d^3 \beta}{dz^3}. \quad (3.10)$$

In terms of Γ , defined as

$$\Gamma = I_f \frac{h^2}{2}, \quad (3.11)$$

Eq. (3.10) can be written as

$$T_r = -E\Gamma \frac{d^3 \beta}{dz^3}. \quad (3.12)$$

We have then, from Eq. (3.1),

$$T = GJ \frac{d\beta}{dz} - E\Gamma \frac{d^3 \beta}{dz^3}. \quad (3.13)$$

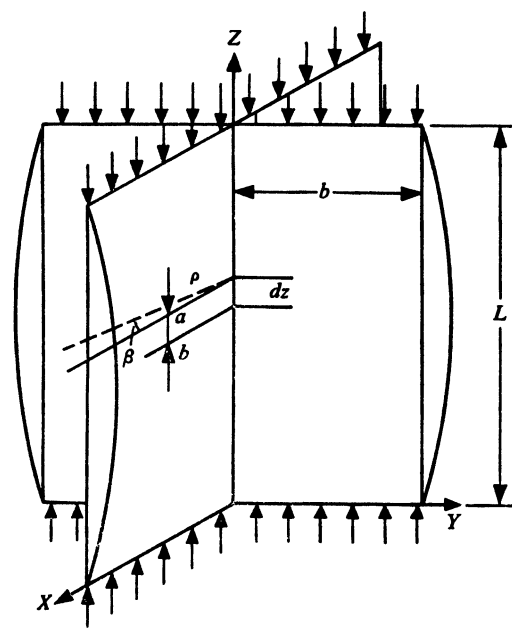
The governing equation of equilibrium for a noncircular section subjected to a torque T at the tip is given by (3.13). For more details, the interested reader may see Chajes [3].

3.3 TORSIONAL BUCKLING

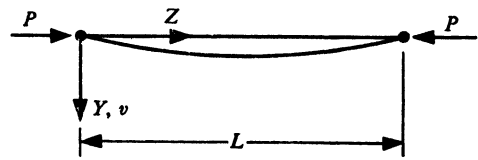
3.3.1 Equilibrium Approach

The problem of torsional buckling has been studied extensively by many including Timoshenko and Gere [4], Hoff [5], and Chajes [3]. We shall discuss some of the methods these investigators employed to obtain the governing equation for torsional buckling.

Consider a doubly symmetric section as shown in Fig. 3.3a. (Such sections as symmetric *I*-section and *Z*-section, where also the shear centre coincides with the centre of gravity of the section, belong to the class of doubly symmetric sections.) Assume that it has equal



(a) Doubly symmetric column under compressive load



(b) Section of bent shape of column

Fig. 3.3 Symmetric Column under Axial Load.

flanges, each of width *h* and thickness *t*. Such a section can have a pure flexural mode in the *XZ*- or *YZ*-plane, depending on the axis about which the second moment of area is minimum. It can also simultaneously have a pure torsional mode. To determine the critical load

at buckling, it is necessary to consider the deflection of the flanges during buckling. This can be done as follows.

Assume that, at buckling, the column has a bent configuration. In this situation, the load P acts on a slightly rotated cross-section. Further, because of the deflection, the stress at any section will be the algebraic sum of the compressive stress and the bending stress. To understand how a torque is generated as a consequence of the applied load, consider a section at a distance ρ from the axis, as shown in Fig. 3.3b. This corresponds to a column under axial load and, as shown in Section 1.4, is equivalent to a beam subjected to a transverse fictitious load of magnitude $-P(d^2v/dz^2)$, where v is the transverse deflection. To evaluate P and v , consider a section ab at distance ρ from the axis and having a cross-sectional area $t \, d\rho$ (Fig. 3.3a). The transverse deflection v as a result of the rotation β (assuming that during buckling the cross-section rotates but does not distort) is given by

$$v = \rho\beta, \quad (3.14)$$

where β is a function of z alone. The compressive forces acting at the ends of the element ab are $\sigma t \, d\rho$. Thus, the equivalent transverse load intensity $q(z)$ is given by

$$q(z) = -\sigma t \, d\rho \frac{d^2v}{dz^2}. \quad (3.15)$$

Substituting for v from Eq. (3.14), we get

$$q(z) = -\sigma t \rho \, d\rho \frac{d^2\beta}{dz^2}. \quad (3.16)$$

This load acts at a distance ρ from the axis of the column, and hence causes a moment about the Z -axis. (This moment is very small if the width of the flanges is small.) Summing up the entire moment due to the cross-section at a distance z along the column axis, we have

$$\begin{aligned} m_z \, dz &= -\sigma \, dz \frac{d^2\beta}{dz^2} \int_A t \rho^2 \, d\rho \\ &= -\sigma I_0 \frac{d^2\beta}{dz^2} \, dz, \end{aligned} \quad (3.17)$$

where I_0 is the polar moment of inertia of the cross-section about the shear centre. Thus, the torque generated per unit length is

$$m_z = -\sigma I_0 \frac{d^2\beta}{dz^2}. \quad (3.18)$$

If we assume that the torsional moment is positive anticlockwise and increases along the positive Z -direction, then the rate of change of torque T along Z is

$$m_z = -dT/dz. \quad (3.19)$$

Differentiating Eq. (3.13) with respect to z once and making use of Eq. (3.18), we get

$$EI \frac{d^4\beta}{dz^4} - (GJ - \sigma I_0) \frac{d^2\beta}{dz^2} = 0. \quad (3.20)$$

Replacing σ by P/A_c , where A_c is the area of the cross-section, we can rewrite Eq. (3.20) as

$$EI \frac{d^4 \beta}{dz^4} - (GJ - \frac{P}{A_c} I_0) \frac{d^2 \beta}{dz^2} = 0. \quad (3.21)$$

Equation (3.21) is the governing equation for the torsional buckling of a column whose centre of gravity coincides with the elastic axis. In deriving this equation, we assumed that the flanges of the column are wide but not wide enough to be treated as individual plate elements. In Chapter 5, we shall analyze the sections that have wide flanges.

As already noted, a column, in addition to the torsional mode, may have pure flexural modes. In such a situation, the buckling load will be the minimum of the three loads (corresponding to the three modes). For a method to obtain the buckling load due to flexure, see Chapter 1.

We shall now illustrate through examples the technique to get the torsional buckling load.

EXAMPLE 3.1 (Torsional buckling of *I*-section column under axial load) Consider a column of symmetrical *I*-section as shown in Fig. 3.4. The governing equation for torsional buckling is given by (3.21). This can be rewritten in a convenient form as

$$\frac{d^4 \beta}{d\xi^4} + \lambda^2 \frac{d^2 \beta}{d\xi^2} = 0, \quad (3.22)$$

where

$$\lambda^2 = (\frac{P}{A_c} I_0 - GJ) L^2 / (EI')$$

and ξ is a nondimensional parameter defined as $z = \xi L$. Equation (3.22) assumes the same form as Eq. (1.61) if, in the latter, v is replaced by β and λ is redefined as just done. The general solution of Eq. (3.22) is

$$\beta = c_1 \sin \lambda \xi + c_2 \cos \lambda \xi + c_3 \xi + c_4. \quad (3.23)$$

The constants c_1 , c_2 , c_3 , and c_4 have to be determined from the boundary conditions. Let us assume that the ends of the column are constrained in a manner that the column cannot rotate about the *Z*-axis, that is,

$$\beta = 0 \quad (\xi = 0, 1). \quad (3.24)$$

If the ends are free to warp, that is, no warping stresses develop, then, from Eq. (3.18),

$$d^2 \beta / dz^2 = 0 \quad (z = 0, L). \quad (3.25)$$

Equation (3.25) in terms of the nondimensional length ξ is

$$d^2 \beta / d\xi^2 = 0 \quad (\xi = 0, 1). \quad (3.26)$$

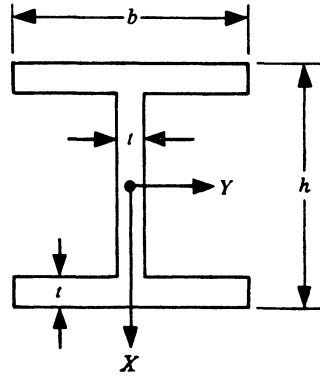


Fig. 3.4 Cross-Section of *I*-Section Column.

Substituting the boundary conditions (3.24) and (3.26) in Eq. (3.23), we get the system of algebraic equations

$$c_2 + c_4 = 0, \quad (3.27a)$$

$$c_1 \sin \lambda + c_2 \cos \lambda + c_3 + c_4 = 0, \quad (3.27b)$$

$$-\lambda^2 c_2 = 0, \quad (3.27c)$$

$$c_1 \lambda^2 \sin \lambda + c_2 \lambda^2 \cos \lambda = 0. \quad (3.27d)$$

Solving for c_1 , c_2 , c_3 , and c_4 , we have

$$c_2 = c_3 = c_4 = 0, \quad (3.28)$$

$$\sin \lambda = 0 \quad \text{if } c_1 \neq 0. \quad (3.29)$$

From Eq. (3.29),

$$\lambda = n\pi. \quad (3.30)$$

Substituting for λ , we obtain

$$P_{cr} = \frac{A_c}{I_0} (GJ + \frac{n^2 \pi^2}{L^2} EI). \quad (3.31)$$

The smallest critical load is obtained when $n = 1$, and this is given by

$$P_{cr1} = \frac{A_c}{I_0} (GJ + \frac{\pi^2}{L^2} EI). \quad (3.32)$$

The mode shape corresponding to this load is of the form

$$\beta = c_1 \sin \pi \xi = c_1 \sin \frac{\pi z}{L}. \quad (3.33)$$

In addition to the load represented by Eq. (3.32), we have the buckling loads due to flexure. Assuming that the ends are hinged, we have

$$P_{cr2} = \pi^2 EI_x / L^2, \quad (3.34)$$

$$P_{cr3} = \pi^2 EI_y / L^2, \quad (3.35)$$

where I_x and I_y are the second moment of area about the X - and Y -axis. The minimum of P_{cr1} , P_{cr2} , and P_{cr3} is taken as the buckling load. As can be seen, the buckling loads are independent of one another.

EXAMPLE 3.2 (Torsional buckling of I -section column under axial load) Consider the same column as in Example 3.1. In addition, assume that this column has built-in ends such that it cannot rotate about the Z -axis and also it cannot warp. The boundary conditions then are

$$\beta = 0 \quad (\xi = 0, 1), \quad (3.36a)$$

$$d\beta/d\xi = 0 \quad (\xi = 0, 1). \quad (3.36b)$$

These conditions correspond to a fixed end column for flexural buckling. The governing equation and the general solution are given by (3.22) and (3.23), respectively. Incorporating the boundary conditions, we get

$$c_2 + c_4 = 0, \quad (3.37a)$$

$$c_1 \sin \lambda + c_2 \cos \lambda + c_3 + c_4 = 0, \quad (3.37b)$$

$$\lambda c_1 + c_3 = 0, \quad (3.37c)$$

$$\lambda c_1 \cos \lambda - c_2 \sin \lambda + c_3 = 0. \quad (3.37d)$$

Equations (3.37) are a set of homogeneous algebraic equations in $c_1, \dots, c_4$. Hence, for a nontrivial solution, the determinant of the coefficients of $c_1, \dots, c_4$ must be zero. Thus,

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ \sin \lambda & \cos \lambda & 1 & 1 \\ \lambda & 0 & 1 & 0 \\ \lambda \cos \lambda & -\lambda \sin \lambda & 1 & 0 \end{vmatrix} = 0. \quad (3.38)$$

Now, the characteristic equation is

$$2(\cos \lambda - 1) + \lambda \sin \lambda = 0. \quad (3.39)$$

Using the trigonometric functions, we can write Eq. (3.39) as

$$\sin \frac{\lambda}{2} \left(\frac{\lambda}{2} \cos \frac{\lambda}{2} - \sin \frac{\lambda}{2} \right) = 0. \quad (3.40)$$

One solution of this equation is

$$\sin \frac{\lambda}{2} = 0,$$

and therefore $\lambda = 2n\pi$ and

$$P_{cr} = \frac{A_c}{I_0} \left(GJ + \frac{4n^2\pi^2}{L^2} EI \right). \quad (3.41)$$

The minimum value of this load is obtained when $n = 1$, and this is given by

$$P_{cr} = \frac{A_c}{I_0} \left(GJ + \frac{4\pi^2}{L^2} EI \right). \quad (3.42)$$

For this load, we find, from Eqs. (3.37), that

$$c_1 = c_3 = 0, \quad c_2 = -c_4$$

and that the buckled shape is given by

$$\beta = c_2(\cos 2n\pi\xi - 1). \quad (3.43)$$

A second solution of Eq. (3.40) is also possible if we set equal to zero the quantity within

the parentheses in this equation. This leads to the transcendental equation

$$\tan \frac{\lambda}{2} = \frac{\lambda}{2}. \quad (3.44)$$

The lowest root of this equation is $\lambda/2 = 4.493$, and therefore

$$P_{cr} = \frac{A}{I_0} \left(GJ + \frac{80.75EI}{L^2} \right). \quad (3.45)$$

This load is higher than that given by Eq. (3.42) and corresponds to the antisymmetric mode of buckling.

3.4 TORSIONAL-FLEXURAL BUCKLING

3.4.1 Equilibrium Approach

Consider a column of general cross-section as shown in Fig. 3.5. Here, X and Y represent the principal axes of the section passing through the centroid O and S shows the location of shear centre of the cross-section. Assume that the distance of S from O is x_0 and y_0 along the X - and Y -axis. This distance, being a sectional property, can be obtained once the dimensions of the section are known. (For details on a method for obtaining the shear centre of any section, see Peery [1].) The total deformation of the section is due to translation and rotation. Assume further (see Section 3.3) that the section does not distort during deformation. In view of the small deformation assumption, the total deformation is a combination of flexure (in which the line OS assumes the position $O'S'$) and rotation about the point S' . The position of the centroid after deformation is given by O'' . Since the section does not distort, the lengths OS , $O'S'$, and $O''S'$ remain constant. Let u and v represent the displacement of the point S along the X - and Y -direction. The displacement of O with respect to the X - and Y -axis is obtained as follows. Let u' and v' be the displacement of O'' with respect to O . From geometry,

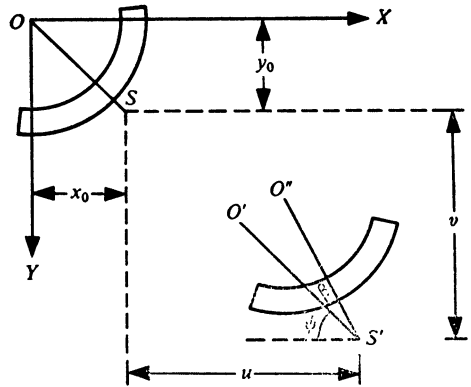


Fig. 3.5 Torsional-Flexural Deformation of Section.

$$\begin{aligned} u' &= x_0 + u - O'S' \cos(\beta + \psi) \\ &= x_0 + u - O'S' \cos \beta \cos \psi + O'S' \sin \beta \sin \psi \\ &= x_0 + u - x_0 \cos \beta + y_0 \sin \beta. \end{aligned} \quad (3.46)$$

For small deformations, $\cos \beta \approx 1$ and $\sin \beta = \beta$. Hence,

$$u' = u + y_0 \beta, \quad (3.47)$$

$$\begin{aligned}
 v' &= y_0 + v - O''S' \sin(\beta + \psi) \\
 &= y_0 + v - O''S' \sin \beta \cos \psi - O''S' \cos \beta \sin \psi \\
 &= y_0 + v - x_0 \sin \beta - y_0 \cos \beta
 \end{aligned} \tag{3.48}$$

$$= v - x_0 \beta. \tag{3.49}$$

Thus, the centroid moves through a distance $(u + y_0\beta)$ and $(v - x_0\beta)$ with respect to the X - and Y -axis.

The differential equations for the deflection curve of the axis are

$$\frac{d^2}{dz^2}(EI_y \frac{d^2u}{dz^2}) + P(\frac{d^2u}{dz^2} + y_0 \frac{d^2\beta}{dz^2}) = 0, \tag{3.50}$$

$$\frac{d^2}{dz^2}(EI_x \frac{d^2v}{dz^2}) + P(\frac{d^2v}{dz^2} - x_0 \frac{d^2\beta}{dz^2}) = 0. \tag{3.51}$$

To obtain the governing equation for β , we shall use the method of Section 3.3. Consider a longitudinal strip of cross-section $t \, ds$, defined by x and y in the plane of the cross-section. The components of its deflection in the X - and Y -direction during buckling are $u + (y_0 - y)\beta$ and $v - (x_0 - x)\beta$. Hence, the intensities of the transverse forces in the X - and Y -direction [see Eq. (3.16)] are

$$-\sigma t \, ds \frac{d^2}{dz^2}[u + (y_0 - y)\beta],$$

$$-\sigma t \, ds \frac{d^2}{dz^2}[v - (x_0 - x)\beta].$$

Taking the moment of these forces about the shear centre axis, we obtain, for one longitudinal strip, the torque per unit length as

$$\begin{aligned}
 m_z &= \int_A [-\sigma t \, ds (y_0 - y) \{ \frac{d^2u}{dz^2} + (y_0 - y) \frac{d^2\beta}{dz^2} \} \\
 &\quad + \sigma t \, ds (x_0 - x) \{ \frac{d^2v}{dz^2} + (x_0 - x) \frac{d^2\beta}{dz^2} \}] \, dA,
 \end{aligned} \tag{3.52}$$

$$\frac{dT}{dz} = -m_z = -[P(x_0 \frac{d^2v}{dz^2} - y_0 \frac{d^2u}{dz^2}) - I_0 \frac{P}{A_c} \frac{d^2\beta}{dz^2}], \tag{3.53}$$

where

$$\sigma \int_A t \, ds = P, \quad \int_A x t \, ds = \int_A y t \, ds = 0,$$

$$\int_A y^2 t \, ds = I_x, \quad \int_A x^2 t \, ds = I_y,$$

$$I_0 = I_x + I_y + A_c(x_0^2 + y_0^2)$$

with I_x and I_y as the second moment of area about the principal axes and I_0 as the polar moment of inertia about the shear centre.

Differentiating Eq. (3.13) with respect to z and substituting for dT/dz from Eq. (3.53), we get

$$EI \frac{d^4 \beta}{dz^4} - (GJ - \frac{I_0}{A_c} P) \frac{d^2 \beta}{dz^2} - P x_0 \frac{d^2 v}{dz^2} + P y_0 \frac{d^2 u}{dz^2} = 0. \quad (3.54)$$

Equations (3.50), (3.51), and (3.54) are three governing equations which are coupled, and hence, for a section where $x_0 \neq 0$ and $y_0 \neq 0$, we have to solve these coupled differential equations. If a section has one axis of symmetry, that is, the section is symmetric about the X -axis, then $y_0 = 0$, and Eq. (3.50) gets uncoupled from Eqs. (3.51) and (3.54) and can be solved separately. If a section is symmetric about the Y -axis, then $x_0 = 0$, and Eq. (3.51) gets uncoupled from Eqs. (3.50) and (3.54). If, however, a section possesses symmetry with respect to the X - and Y -axis, all the three equations then get uncoupled and can be solved independently. We have already discussed this situation (see Section 3.3).

We shall now consider examples of the coupled torsion-flexure mode.

EXAMPLE 3.3 (Buckling of channel section with hinged ends) Consider a cross-section of a channel section of a uniform column as shown in Fig. 3.6. The section possesses one

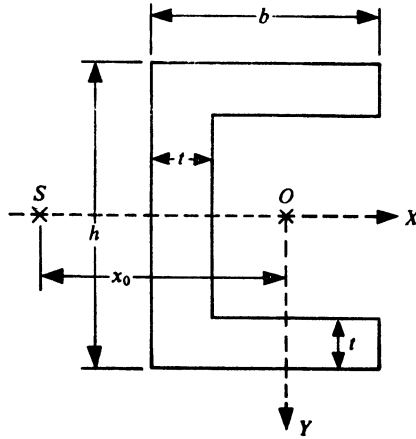


Fig. 3.6 Cross-Section of Channel Section Column.

axis of symmetry about the X -axis, and hence $y_0 = 0$. In view of this, the governing equations (3.50), (3.51), and (3.54) reduce to

$$EI_y \frac{d^4 u}{dz^4} + P \frac{d^2 u}{dz^2} = 0, \quad (3.55)$$

$$EI_x \frac{d^4 v}{dz^4} + P \frac{d^2 v}{dz^2} - P(-x_0) \frac{d^2 \beta}{dz^2} = 0, \quad (3.56)$$

$$EI \frac{d^4 \beta}{dz^4} - (GJ - \frac{I_0}{A_c} P) \frac{d^2 \beta}{dz^2} - P(-x_0) \frac{d^2 v}{dz^2} = 0. \quad (3.57)$$

In terms of the nondimensional variable ξ , defined as $\xi = z/L$, Eqs. (3.55) to (3.57) can be rewritten as

$$EI_y \frac{d^4 u}{d\xi^4} + PL^2 \frac{d^2 u}{d\xi^2} = 0, \quad (3.58)$$

$$EI_x \frac{d^4 v}{d\xi^4} + PL^2 \frac{d^2 v}{d\xi^2} + Px_0 L^2 \frac{d^2 \beta}{d\xi^2} = 0, \quad (3.59)$$

$$EI \frac{d^4 \beta}{d\xi^4} - (GJ - \frac{I_0}{A_c} P) L^2 \frac{d^2 \beta}{d\xi^2} + Px_0 L^2 \frac{d^2 v}{d\xi^2} = 0. \quad (3.60)$$

Equation (3.58) is independent of β , and hence the buckling is independent of torsion and the mode of form failure in the plane of symmetry is flexural. On the other hand, Eqs. (3.59) and (3.60) are functions of β , and hence the buckling perpendicular to the plane of symmetry is coupled with torsion. To solve any of these equations, we should know I_y , I_x , Γ , J , and I_0 . For the given section,

$$J = \frac{2}{3}bt^3 + \frac{1}{3}ht^3 \quad \text{from Eq. (3.3),}$$

$$\Gamma = \frac{th^2b^3}{12} \left(\frac{2}{1} + \frac{3b/h}{6b/h} \right) \quad \text{from Burge [6],}$$

$$x_0 = b \left(\frac{3b/h}{1 + 6b/h} + \frac{b/h}{1 + 2b/h} \right) \quad \text{from Burge [6],}$$

$$I_0 = I_x + I_y + (2bt + ht)x_0^2.$$

We now assume that the ends are hinged such that the column can rotate, but cannot deflect, about the X - and Y -axis. Further, the column is free to warp, but cannot rotate, about the Z -axis. The boundary conditions then are

$$u = 0, \quad \frac{d^2 u}{d\xi^2} = 0 \quad (\xi = 0, 1), \quad (3.61)$$

$$v = 0, \quad \frac{d^2 v}{d\xi^2} = 0 \quad (\xi = 0, 1), \quad (3.62)$$

$$\beta = 0, \quad \frac{d^2 \beta}{d\xi^2} = 0 \quad (\xi = 0, 1). \quad (3.63)$$

Since Eqs. (3.59) and (3.60) are a set of simultaneous homogeneous differential equations of even order with constant coefficients, we can look for a variable separable type of solution. This implies that the mode shapes in v and β do not change because of coupling or, in other words, the changes in the mode shapes in v and β because of coupling are of the second order, and hence can be neglected.

The shape functions which satisfy the boundary conditions (3.61), (3.62), and (3.63) are

$$u = A \sin \pi \xi, \quad v = B \sin \pi \xi, \quad \beta = C \sin \pi \xi, \quad (3.64)$$

where A , B , and C are arbitrary constants and $\sin \pi \xi$ represents the first mode shape of

buckling both in flexure and torsion. Now, Eq. (3.55) can be solved independently by substituting in it the first of Eqs. (3.64). This gives

$$(EI_y\pi^4 - P_{cr}L^2\pi^2)A \sin \pi\xi = 0. \quad (3.65)$$

Since $A \sin \pi\xi \neq 0$, for Eq. (3.65) to be satisfied, the quantity within the parentheses must be zero. This leads to

$$P_{cr} = \pi^2 EI_y / L^2. \quad (3.66)$$

Substituting the second and the third of Eqs. (3.64) in each of Eqs. (3.59) and (3.60), we get

$$(EI_x\pi^4 - PL^2\pi^2)B - x_0L^2\pi^2PC = 0, \quad (3.67)$$

$$-x_0L^2\pi^2PB + [E\Gamma\pi^4 + (GJ - \frac{I_0}{A_0}P)\pi^2L^2]C = 0. \quad (3.68)$$

For a nontrivial solution of Eqs. (3.67) and (3.68), the determinant of the coefficients of B and C must be zero. Thus,

$$\begin{vmatrix} EI_x\pi^4 - PL^2\pi^2 & -x_0L^2\pi^2P \\ -x_0L^2\pi^2P & E\Gamma\pi^4 + (GJ - \frac{I_0}{A_c}P)\pi^2L^2 \end{vmatrix} = 0. \quad (3.69)$$

Now, the characteristic equation is

$$P^2(1 - \frac{x_0^2}{I_0}A_c) - (P_x + P_\beta)P + P_xP_\beta = 0, \quad (3.70)$$

where

$$P_x = \frac{\pi^2 EI_x}{L^2}, \quad P_\beta = \frac{A_c}{I_0}(GJ + \frac{\pi^2}{L^2}E\Gamma).$$

Equation (3.70) is a quadratic equation, and hence has two roots, namely, P_{cr1} and P_{cr2} , of which one, i.e., P_{cr1} , is smaller than P_x or P_β , whereas P_{cr2} is larger than either of them. Therefore, the buckling load will be the smaller of P_{cr1} and P_{cr} [see Eq. (3.66)].

EXAMPLE 3.4 (Buckling of channel section with clamped ends) Consider, once again, Example 3.3 this time with clamped boundary conditions. The governing equations remain the same and are given by (3.58), (3.59), and (3.60). On the other hand, the boundary conditions for the deformations, in terms of the nondimensional variable ξ , about the X - and Y -axis now become

$$u = 0, \quad \frac{du}{d\xi} = 0 \quad (\xi = 0, 1), \quad (3.71)$$

$$v = 0, \quad \frac{dv}{d\xi} = 0 \quad (\xi = 0, 1). \quad (3.72)$$

For the torsional mode, the boundary conditions are such that the column is restrained

from warping and also cannot rotate about the Z -axis. This can be mathematically expressed as

$$\beta = 0, \quad \frac{d\beta}{d\xi} = 0 \quad (\xi = 0, 1). \quad (3.73)$$

The mode shapes which satisfy the boundary conditions (3.71) to (3.73) are

$$\begin{aligned} u &= A(1 - \cos 2\pi\xi), \\ v &= B(1 - \cos 2\pi\xi), \\ \beta &= C(1 - \cos 2\pi\xi). \end{aligned} \quad (3.74)$$

Substituting the first of Eqs. (3.74) in Eq. (3.58), we get

$$[-EI_y(2\pi)^4 + P_{cr}L^2(2\pi)^2]A \cos 2\pi\xi = 0. \quad (3.75)$$

Since $A \cos 2\pi\xi \neq 0$, the quantity within the square brackets must be zero. This yields the P_{cr} for buckling in the plane of symmetry as

$$P_{cr} = 4\pi^2 EI_y / L^2. \quad (3.76)$$

Substituting the second and the third of Eqs. (3.74) in each of Eqs. (3.59) and (3.60), we get

$$[-EI_x(2\pi)^4 + PL^2(2\pi)^2]B + P_{x0}L^2(2\pi)^2C = 0, \quad (3.77)$$

$$P_{x0}L^2(2\pi)^2B + [-EI(2\pi)^4 - (GJ - \frac{I_0}{A_c}P)(2\pi)^2L^2]C = 0. \quad (3.78)$$

Following the same procedure as in Example 3.3, the characteristic equation we obtain is

$$P^2(1 - \frac{x_0^2}{I_0}A) - (P_x + P_\beta)P + P_xP_\beta = 0. \quad (3.79)$$

For the boundary conditions (3.71) to (3.73), P_x and P_β are given by

$$\begin{aligned} P_x &= 4\pi^2 EI_x / L^2, \\ P_\beta &= \frac{A_c}{I_0}(GJ + \frac{4\pi^2}{L^2}EI). \end{aligned}$$

Knowing P_x and P_β , we can evaluate the roots P_{cr1} and P_{cr2} of the quadratic equation (3.79). The first buckling load for the column will then be the minimum of P_{cr1} and P_{cr2} [see Eq. (3.76)]. If P_{cr} is minimum, we have a pure flexural mode; on the other hand, if P_{cr1} is minimum, we have a torsion-flexure mode.

EXAMPLE 3.5 (Buckling of unsymmetric section) Consider now a general case where the section does not possess any symmetry. Let us assume that the ends of the column are hinged such that the ends can rotate about the X - and Y -axis but cannot rotate about the Z -axis or deflect in the X - and Y -direction. Further, the ends are free to warp.

For the situation described, the boundary conditions are

$$u = \frac{d^2u}{d\xi^2} = 0 \quad (\xi = 0, 1), \quad (3.80)$$

$$v = \frac{d^2v}{d\xi^2} = 0 \quad (\xi = 0, 1), \quad (3.81)$$

$$\beta = \frac{d^2\beta}{d\xi^2} = 0 \quad (\xi = 0, 1). \quad (3.82)$$

For a general section, x_0 and y_0 both exist. Also, the governing equations in terms of ξ are

$$EI_y \frac{d^4u}{d\xi^4} + PL^2 \frac{d^2u}{d\xi^2} + Py_0 L^2 \frac{d^2\beta}{d\xi^2} = 0, \quad (3.83)$$

$$EI_x \frac{d^4v}{d\xi^4} + PL^2 \frac{d^2v}{d\xi^2} - Px_0 L^2 \frac{d^2\beta}{d\xi^2} = 0, \quad (3.84)$$

$$EI \frac{d^4\beta}{d\xi^4} - (GJ - \frac{I_0}{A_c} P) L^2 \frac{d^2\beta}{d\xi^2} - Px_0 L^2 \frac{d^2v}{d\xi^2} + Py_0 L^2 \frac{d^2u}{d\xi^2} = 0. \quad (3.85)$$

A set of functions of the type

$$u = A \sin \pi \xi, \quad v = B \sin \pi \xi, \quad \beta = C \sin \pi \xi \quad (3.86)$$

satisfies the boundary conditions (3.80) to (3.82) exactly. Substituting expressions (3.86) in Eqs. (3.83) to (3.85), we get

$$(EI_y \pi^4 - PL^2 \pi^2) A - L^2 Py_0 \pi^2 C = 0, \quad (3.87)$$

$$(EI_x \pi^4 - PL^2 \pi^2) B + L^2 Px_0 \pi^2 C = 0, \quad (3.88)$$

$$[EI \pi^4 + (GJ - \frac{I_0}{A_c} P) \pi^2 L^2] C + L^2 Px_0 \pi^2 B - Py_0 L^2 \pi^2 A = 0. \quad (3.89)$$

Equations (3.87) to (3.89) are a set of simultaneous homogeneous algebraic equations, and hence, for a nontrivial solution of A , B , and C , the determinant of the coefficients of A , B , and C must vanish. This determinant is given by

$$\begin{vmatrix} (P - P_y) & 0 & Py_0 \\ 0 & (P - P_x) & -Px_0 \\ Py_0 & -Px_0 & \frac{I_0}{A_c}(P - P_\beta) \end{vmatrix}. \quad (3.90)$$

Setting the determinant equal to zero, we get the characteristic equation as

$$P^3 \left[1 - \frac{A_c}{I_0} (x_0^2 + y_0^2) \right] + P^2 \left[\frac{A_c}{I_0} (P_x y_0^2 + P_y x_0^2) - (P_x + P_y + P_\beta) \right] \\ + P(P_x P_y + P_x P_\beta + P_y P_\beta) - P_x P_y P_\beta = 0. \quad (3.91)$$

Knowing the dimensions of the cross-section and the column material property, we can evaluate the three roots of Eq. (3.91). The smallest of these roots gives the buckling load for the cross-section.

Timoshenko and Gere [4] have further analyzed Eq. (3.91), assuming relative magnitudes of P_x , P_y , and P_β . According to them, if $P_x < P_y$, then this equation yields three positive real roots, of which one is less than P_x , one greater than P_y , and the third between P_x and P_y . A similar observation can be made when $P_x > P_y$. It can also be shown that, of these three roots, the smallest is less than, and the largest greater than, P_β . Thus, one critical load is less, and the other greater, than P_x , P_y , or P_β . The third root, as already noted, always lies between P_x and P_y . As stated in Section 3.3, for wide flange open sections, the smallest root is closer to P_β .

3.5 ENERGY APPROACH

The differential equation approach can be successfully used for a coupled mode if the cross-section is uniform throughout the axis and the boundary conditions are symmetric. If any of these conditions is violated, this approach ceases to be applicable. In such a situation, the energy approach, which has the inherent ability to converge to the exact solution, can be employed to obtain an approximate solution. This requires (see Section 1.2.2) the calculation of total potential of the system defined as

$$\Pi_p = U + V_e = \text{constant.} \quad (1.17)$$

The strain energy U consists of the energy due to flexure about the X - and Y -axis and that due to torsion about the Z -axis. The energy due to torque is composed of the (i) energy associated with the St. Venant torque and (ii) energy due to the axial constraint stresses (because the warping is prevented).

The energy due to flexure is given by Eq. (1.22) and can be written as

$$U_b = \frac{1}{2} \int_0^L EI_y \left(\frac{d^2 u}{dz^2} \right)^2 dz + \frac{1}{2} \int_0^L EI_x \left(\frac{d^2 v}{dz^2} \right)^2 dz. \quad (3.92)$$

The energy due to torque T is obtained as follows.

Contribution from St. Venant torque

According to Eq. (3.2),

$$d\beta = \frac{T}{GJ} dz.$$

Hence,

$$\begin{aligned} U_{st} &= \frac{1}{2} \int_0^L \int_A \tau \gamma \, dz \, dA \\ &= \frac{1}{2} \int_0^L \int_A \frac{\tau^2}{G} \, dz \, dA. \end{aligned} \quad (3.93)$$

Substituting for τ from Peery [1], we get

$$\begin{aligned} U_{st} &= \frac{1}{2} \int_0^L \int_A \frac{T^2 r^2}{J^2 G} dA dz \\ &= \frac{1}{2} \int_0^L \frac{T^2}{GJ} dz. \end{aligned} \quad (3.94)$$

Substituting for T from Eq. (3.2) in Eq. (3.94), we get

$$U_{st} = \frac{1}{2} \int_0^L GJ \left(\frac{d\beta}{dz} \right)^2 dz. \quad (3.95)$$

Contribution from axial constraint stresses

We shall consider only the bending energy due to the axial constraint stresses. The bending energy is

$$U_{tb} = \frac{1}{2} \int_0^L M_f \left(\frac{d^2 v}{dz^2} \right)^2 dz. \quad (3.96)$$

Substituting for M_f and v from Eqs. (3.7) and (3.8) in Eq. (3.96), we get

$$U_{tb} = \frac{1}{2} \int_0^L EI_f \frac{h^2}{4} \left(\frac{d^2 \beta}{dz^2} \right)^2 dz. \quad (3.97)$$

In view of relation (3.11),

$$U_{tb} = \frac{1}{2} \int_0^L E \frac{\Gamma}{2} \left(\frac{d^2 \beta}{dz^2} \right)^2 dz. \quad (3.98)$$

Equation (3.98) gives the bending energy due to one flange alone. Hence, the bending energy due to both the flanges is

$$U_{tb} = \frac{1}{2} \int_0^L E \Gamma \left(\frac{d^2 \beta}{dz^2} \right)^2 dz. \quad (3.99)$$

In terms of the nondimensional variable ξ , Eqs. (3.92), (3.95), and (3.99), when combined, can be written as

$$U = \frac{1}{2} \int_0^1 \frac{EI_y}{L^3} \left(\frac{d^2 u}{d\xi^2} \right)^2 d\xi + \frac{1}{2} \int_0^1 \frac{EI_x}{L^3} \left(\frac{d^2 v}{d\xi^2} \right)^2 d\xi + \frac{1}{2} \int_0^1 \frac{GJ}{L} \left(\frac{d\beta}{d\xi} \right)^2 d\xi + \frac{1}{2} \int_0^1 \frac{E\Gamma}{L^3} \left(\frac{d^2 \beta}{d\xi^2} \right)^2 d\xi. \quad (3.100)$$

The potential energy V_e is obtained by the method we discussed in Section 1.2.2. However, here we have to consider the new position of the centre of gravity with respect to both the X - and Y -axis. This new position for these two axes is given by Eqs. (3.46) and (3.48). Following the procedure given in Section 1.2.2, we get

$$V_e = -\frac{1}{2} \int_0^L \int_A \sigma \left[\left\{ \frac{d}{dz} [u + (y_0 - y)\beta] \right\}^2 + \left\{ \frac{d}{dz} [v - (x_0 - x)\beta] \right\}^2 \right] dA dz. \quad (3.101)$$

After manipulation, we obtain

$$V_e = -\frac{1}{2} \int_0^L \int_A \sigma \left[\left(\frac{du}{dz} \right)^2 + \left(\frac{dv}{dz} \right)^2 + y_0^2 \left(\frac{d\beta}{dz} \right)^2 + x_0^2 \left(\frac{d\beta}{dz} \right)^2 + 2y_0 \frac{du}{dz} \frac{d\beta}{dz} - 2x_0 \frac{dv}{dz} \frac{d\beta}{dz} \right. \\ \left. + y^2 \left(\frac{d\beta}{dz} \right)^2 + x^2 \left(\frac{d\beta}{dz} \right)^2 - 2yy_0 \left(\frac{d\beta}{dz} \right)^2 - 2xx_0 \left(\frac{d\beta}{dz} \right)^2 - 2y \frac{d\beta}{dz} \frac{du}{dz} + 2x \frac{d\beta}{dz} \frac{dv}{dz} \right] dA dz \quad (3.102)$$

$$= -\frac{P}{2} \int_0^L \left[\left(\frac{du}{dz} \right)^2 + \left(\frac{dv}{dz} \right)^2 + \frac{I_0}{A_c} \left(\frac{d\beta}{dz} \right)^2 - 2x_0 \frac{dv}{dz} \frac{d\beta}{dz} + 2y_0 \frac{du}{dz} \frac{d\beta}{dz} \right] dz, \quad (3.103)$$

where

$$I_0 = I_x + I_y + A_c(x_0^2 + y_0^2).$$

Combining Eqs. (3.100) and (3.103), we can get the total potential of the system.

3.5.1 Rayleigh-Ritz Method

The problem of finding the critical load is equivalent to that of finding the deflected shape for which a system is in equilibrium. This can be achieved by making the first variation of the total potential to vanish. This is stated by Eq. (1.16).

We shall consider here examples to illustrate the Rayleigh-Ritz technique.

EXAMPLE 3.6 (Buckling of channel section with hinged ends) Consider Example 3.3. The requirements for the shape functions are that they must satisfy the geometric boundary conditions. The geometric boundary conditions for this column are

$$\begin{aligned} u &= 0 & (z = 0, L), \\ v &= 0 & (z = 0, L), \\ \beta &= 0 & (z = 0, L). \end{aligned} \quad (3.104)$$

In terms of the nondimensional parameter ξ , U is given by Eq. (3.100). However, V_e is given by

$$V_e = -\frac{P}{2L} \int_0^1 \left[\left(\frac{du}{d\xi} \right)^2 + \left(\frac{dv}{d\xi} \right)^2 + \frac{I_0}{A_c} \left(\frac{d\beta}{dz} \right)^2 - 2x_0 \frac{dv}{d\xi} \frac{d\beta}{dz} + 2y_0 \frac{du}{d\xi} \frac{d\beta}{dz} \right] d\xi. \quad (3.105)$$

Equation (3.105) can be simplified by using the symmetry of the structure. Here, $x_0 = -x_0$, $y_0 = 0$. Substituting these in Eq. (3.105), we get

$$V_e = -\frac{P}{2L} \int_0^1 \left[\left(\frac{du}{d\xi} \right)^2 + \left(\frac{dv}{d\xi} \right)^2 + \frac{I_0}{A_c} \left(\frac{d\beta}{dz} \right)^2 + 2x_0 \frac{dv}{d\xi} \frac{d\beta}{dz} \right] d\xi. \quad (3.106)$$

Clearly, whatever be the cross-section, the expression for U remains the same; however, for V_e , it changes.

The geometric boundary conditions are identically satisfied by assuming a deflected shape of the form

$$\begin{aligned}
u &= A \sin \pi \xi, \\
v &= B \sin \pi \xi, \\
\beta &= C \sin \pi \xi.
\end{aligned} \tag{3.107}$$

Substituting Eqs. (3.107) in Eqs. (3.100) and (3.106) and integrating, we get

$$U = \frac{1}{2} \frac{\pi^2}{L} \left[A^2 \frac{EI_y \pi^2}{L^2} + B^2 \frac{EI_x \pi^2}{L^2} + C^2 \left(GJ + \frac{E\Gamma \pi^2}{L^2} \right) \right], \tag{3.108}$$

$$V_e = -\frac{P}{4L} (A^2 \pi^2 + B^2 \pi^2 + x_0^2 C^2 \pi^2 + 2x_0 BC \pi^2). \tag{3.109}$$

Combining Eqs. (3.108) and (3.109), we finally obtain the total potential as

$$\Pi_p = \frac{1}{2} \frac{\pi^2}{L} \left[A^2 \frac{EI_y \pi^2}{L^2} + B^2 \frac{EI_x \pi^2}{L^2} + C^2 \left(GJ + \frac{E\Gamma \pi^2}{L^2} \right) \right] - \frac{P}{4L} [A^2 \pi^2 + B^2 \pi^2 + C^2 P \frac{I_0}{A_c} \pi^2 + 2BC \pi^2 x_0]. \tag{3.110}$$

Taking the derivatives of expression (3.110) with respect to A , B , and C and setting them equal to zero, we have

$$\frac{\partial \Pi_p}{\partial A} = \left(\frac{EI_y \pi^2}{L^2} - P \right) A = 0, \tag{3.111}$$

$$\frac{\partial \Pi_p}{\partial B} = \left(\frac{EI_x \pi^2}{L^2} - P \right) B - Px_0 C = 0, \tag{3.112}$$

$$\frac{\partial \Pi_p}{\partial C} = -Bx_0 P + \left[\left(GJ + \frac{E\Gamma \pi^2}{L^2} \right) - P \frac{I_0}{A_c} \right] C = 0. \tag{3.113}$$

Clearly, Eq. (3.111) is not coupled to Eqs. (3.112) and (3.113), and hence, if $A \neq 0$,

$$P_{cr} = \pi^2 EI_y / L^2. \tag{3.114}$$

This is the same as Eq. (3.66). Equations (3.112) and (3.113) are coupled algebraic equations, and hence, for a nontrivial solution of B and C , the determinant of the coefficients of B and C must be zero, that is,

$$\begin{vmatrix}
(P_x - P) & -Px_0 \\
-Px_0 & (P_\beta - P) \frac{I_0}{A_c}
\end{vmatrix} = 0. \tag{3.115}$$

The characteristic equation is now given by

$$P^2 \left(1 - \frac{x_0^2}{I_0} A_c \right) - (P_x + P_\beta) P + P_x P_\beta = 0, \tag{3.116}$$

where

$$\begin{aligned}
P_x &= \pi^2 EI_x / L^2, \\
P_\beta &= \frac{A_c}{I_0} \left(GJ + \frac{\pi^2}{L^2} E\Gamma \right).
\end{aligned}$$

Equation (3.116) is a quadratic equation and has two roots, namely, P_{cr1} and P_{cr2} , of which one, i.e., P_{cr1} , is smaller than P_x or P_β , whereas P_{cr2} is larger than either of them. The critical loads obtained here are exactly the same as those obtained by using the differential equation approach. We should have expected this since the shape function we chose satisfies both the geometric and natural boundary conditions and also happens to be the exact solution of the differential equations (3.50), (3.51), and (3.54).

EXAMPLE 3.7 (Buckling of channel section with one end clamped and other free) Since the boundary conditions for the column are not symmetric, it is not possible to obtain the buckling load using the differential equation approach. The geometric boundary conditions are

$$u = 0, \quad \frac{du}{d\xi} = 0 \quad (\xi = 0), \quad (3.117a)$$

$$v = 0, \quad \frac{dv}{d\xi} = 0 \quad (\xi = 0), \quad (3.117b)$$

$$\beta = 0, \quad \frac{d\beta}{d\xi} = 0 \quad (\xi = 0). \quad (3.117c)$$

Let us now choose the shape function for u , v , and β as

$$u = A\xi^2, \quad (3.118a)$$

$$v = B\xi^2, \quad (3.118b)$$

$$\beta = C\xi^2. \quad (3.118c)$$

It can easily be seen that this representation satisfies the geometric boundary conditions. Substituting for u , v , and β from Eqs. (3.118) in Eqs. (3.100) and (3.106) and integrating, we get

$$U = \frac{1}{2} \frac{EI_y}{L^3} 4A^2 + \frac{1}{2} \frac{EI_x}{L^3} 4B^2 + \frac{1}{2} \frac{GJ}{L} \frac{4C^2}{3} + \frac{1}{2} \frac{E\Gamma}{L^2} 4C^2, \quad (3.119)$$

$$V_c = -\frac{P}{2L} \frac{4A^2}{3} + \frac{4B^2}{3} + \frac{4C^2}{3} \frac{I_0}{A_c} + \frac{8}{3} BCx_0. \quad (3.120)$$

The total potential Π_p can now be written as

$$\Pi_p = \frac{1}{2} \left[\frac{EI_y}{L^3} 4A^2 + \frac{EI_x}{L^3} 4B^2 + \frac{GJ}{L} \frac{4C^2}{3} + \frac{E\Gamma}{L^2} 4C^2 \right] - \frac{P}{2L} \left[\frac{4}{3} A^2 + \frac{4}{3} B^2 + \frac{4}{3} C^2 \frac{I_0}{A_c} + \frac{8}{3} BCx_0 \right]. \quad (3.121)$$

Taking the derivatives of expression (3.121) with respect to A , B , and C and setting them equal to zero, we get

$$\frac{\partial \Pi_p}{\partial A} = \left(\frac{3EI_y}{L^2} - P \right) A = 0, \quad (3.122)$$

$$\frac{\partial \Pi_p}{\partial B} = \left(\frac{3EI_x}{L^2} - P \right) B - Px_0 C = 0, \quad (3.123)$$

$$\frac{\partial \Pi_p}{\partial C} = -Px_0 B + \frac{I_0 A_c}{I_0} \left(GJ + \frac{3E\Gamma}{L^2} \right) - P C = 0. \quad (3.124)$$

Equation (3.122) is not coupled to Eqs. (3.123) and (3.124) and gives

$$P_{cr} = 3EI_y/L^2. \quad (3.125)$$

This is 21.3% higher than the exact value for pure flexural buckling [see Eq. (1.85)]. Equations (3.123) and (3.124) yield the characteristic equation

$$P^2 \left(1 - \frac{x_0^2}{I_0} A_c \right) - (P_x + P_\beta) P + P_x P_\beta = 0, \quad (3.126)$$

where

$$P_x = 3EI_x/L^2,$$

$$P_\beta = \frac{A_c}{I_0} \left(GJ + \frac{\pi^2 E \Gamma}{L^2} \right).$$

A better approximation to the exact solution can be obtained by considering shape functions of the type

$$u = A\xi^2(3L - \xi),$$

$$v = B\xi^2(3L - \xi),$$

$$\beta = C\xi^2(3L - \xi).$$

PROBLEMS

3.1 Obtain the critical load for a column of *I*-section (see Fig. 3.7) if the column ends are hinged such that the ends can rotate, but cannot deflect, about the *X*- and *Y*-axis. Further, the ends are free to warp, but cannot rotate, about the *Z*-axis.

3.2 Repeat Problem 3.1, for clamped end conditions, such that the ends cannot rotate or deflect about the *X*- and *Y*-axis. Further, the warping is constrained and the ends cannot rotate about the *Z*-axis.

3.3 Obtain an expression for Γ and J for the section shown in Fig. 3.8.

3.4 For the column in Problem 3.1, estimate the buckling load by using the Rayleigh-Ritz technique. Assume the shape function for u , v , and β as

$$u = A(1 - \xi)\xi,$$

$$v = B(1 - \xi)\xi,$$

$$\beta = C(1 - \xi)\xi.$$

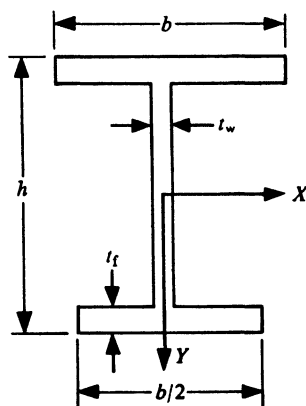


Fig. 3.7 Problem 3.1.

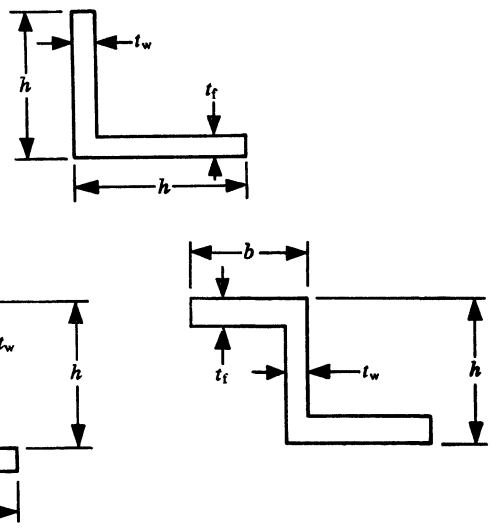


Fig. 3.8 Problem 3.3.

Show that a better estimate can be obtained by choosing functions of the type

$$u = A(\xi^4 - 2\xi^3 + \xi),$$
$$v = B(\xi^4 - 2\xi^3 + \xi),$$
$$\beta = C(\xi^4 - 2\xi^3 + \xi).$$

3.5 Obtain the critical load for the column in Problem 3.2 by considering the following

two-term representation for u , v , and β :

$$u = A_1(1 - \cos 2\pi\xi) + A_2(1 - \cos 4\pi\xi),$$

$$v = B_1(1 - \cos 2\pi\xi) + B_2(1 - \cos 4\pi\xi),$$

$$\beta = C_1(1 - \cos 2\pi\xi) + C_2(1 - \cos 4\pi\xi).$$

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4

Buckling of Frames

4.1 INTRODUCTION

Frames are extensively used in engineering design. For example, a multistoreyed building has to have frames. Further, an overhead tank is constructed on frames. Moreover, in a propeller driven aircraft, the engines are mounted on space frames attached to the wings. In addition, a lathe or a drilling machine is made out of a frame type of structure. A frame thus used is subjected to different kinds of loads which can be broadly classified as static and dynamic loads. A designer has therefore to ensure that a structure does not fail when it is subjected to these loads.

Columns, beams, and beam columns do not occur in isolation but are, instead, joined together to make up a structural frame. Such a frame is a skeletal structure which supports the loads the total structure is called upon to withstand. In Chapters 1 and 2, we examined the stability of each member in a structure separately. In this chapter, we shall consider the response of the frames to those loads that result in the development of compressive stresses.

In a frame, generally, the columns are subjected to bending in addition to the axial forces and the beams are subjected to the axial forces. The buckled mode of a frame member and its *effective length*—a parameter so useful in design—depend on the end conditions of the member. In a rigid jointed frame, on the other hand, the end conditions on a member are determined from the relative stiffnesses of the members meeting at a joint and the member. In other words, the member is assumed to be supported by a set of translation and rotation springs. This spring stiffness affects the buckling characteristics of the member. Thus, an infinite rotational spring stiffness indicates a clamped end, and the zero rotational spring stiffness indicates a pin end. The rigid joints between the members of a frame result in a distortion of the members even when only one member deflects due to buckling. It is therefore necessary that, when analyzing a frame, the actual buckling condition of the total frame be considered.

The study of the stability of frames dates back to 1909. For a description on the methods the various early workers used to investigate the frame stability, see Bleich [1]. These methods can be classified into the following three groups:

- (i) The equilibrium equation approach. In this, the equations are derived in terms of joint displacements, rotations, and interrelation of axial forces.
- (ii) The energy approach.

(iii) The modified moment distribution approach. Here, the moment distribution is so modified that it includes the bending effects due to axial forces.

In what follows, we shall consider both the triangulated and rigid jointed frames. Our discussion, however, would be limited to investigating the stability within the plane.

4.2 TRIANGULATED FRAMES

For a pin jointed statically determinate triangulated frame, the analysis is quite simple since, in such a frame, the axial loads are directly proportional to the applied loads. Here, the initiation of buckling of a member is due to these axial loads. In many practical situations, a rigid jointed triangular frame is employed. In such a frame, as already stated, the members interact with one another. As a result, the members undergo rotation and sway. We shall use the equilibrium approach to study the stability of this type of frame. To do this, it is necessary to obtain the stability functions for each member. For the derivation of stability functions for simple members with different loading and boundary conditions, see Horne and Merchant [2].

We shall obtain the stability functions for the members of a triangulated frame. Figure 4.1 shows such a frame of prismatic members of uniform cross-section. Since the ends A and C are fixed, the members AB and BC can resist bending loads in addition to the axial loads.

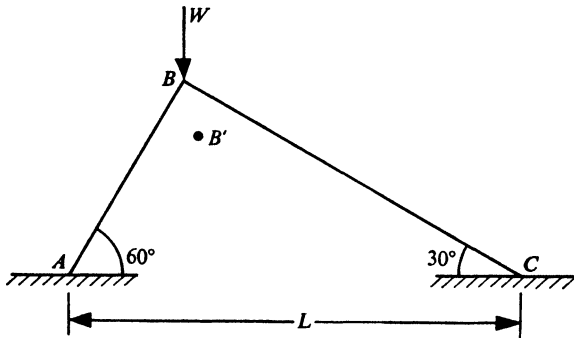


Fig. 4.1 Rigid Jointed Triangular Frame.

It is assumed that the bending loads are quite small and do not alter the axial forces in the members. Hence, the axial forces can be obtained by treating the frame as pin jointed at A , B , and C .

Isolating the joint B , we find the equilibrium equations of this joint are

$$P_{AB} \cos 30^\circ + P_{BC} \cos 60^\circ = W, \quad (4.1)$$

$$P_{AB} \sin 30^\circ + P_{BC} \sin 60^\circ = 0. \quad (4.2)$$

Solving, we get

$$P_{BC} = \frac{W}{2}, \quad P_{AB} = \sqrt{3} \frac{W}{2}. \quad (4.3)$$

These loads cause axial deformation in the members, which is given by

$$\delta_{BC} = \frac{\sqrt{3}}{4} \frac{WL}{AE} = \delta_{AB}. \quad (4.4)$$

The new position of the point B is as indicated by B' (shown exaggerated). Suppose that the frame is moved to this position without rotating the joint B . This results in moments in the members AB and BC . An arbitrary rotation of the joint B would have also resulted in moments. For equilibrium, the net moment at B must be zero. That is,

$$M_{BA} + M_{BC} = 0. \quad (4.5)$$

The total moment M_{BA} is due to the displacement and the rotation of the point B . Thus (see Horne and Merchant [2]),

$$M_{BA} = -\xi_1(1 + \alpha_1) \frac{EI}{L} \frac{\delta_{AB}}{L} + \frac{2}{\sqrt{3}} \frac{EI}{L} \xi_1 \theta, \quad (4.6)$$

$$M_{BC} = 4\xi_2(1 + \alpha_2) \frac{EI}{L} \frac{\delta_{BC}}{L} + 2 \frac{EI}{L} \xi_2 \theta, \quad (4.7)$$

where ξ_1, ξ_2 are the fixity coefficients and α_1, α_2 are the carry-over factors. For a beam type structure, with one end pinned and the other clamped, $\xi_1=4.0$ and $\alpha_1=0.5$. From Eq. (4.5), we have, after substituting for M_{BA} and M_{BC} ,

$$\frac{2EI}{L} \theta \left(\frac{\xi_1}{\sqrt{3}} + \xi_2 \right) = \frac{4EI\delta_{AB}}{L^2} \left[\xi_1 \frac{(1 + \alpha_1)}{3} - \xi_2(1 + \alpha_2) \right]. \quad (4.8)$$

Substituting for δ_{AB} from relation (4.4), we get

$$2 \left(\frac{\xi_1}{\sqrt{3}} + \xi_2 \right) \theta = \sqrt{3} \frac{W}{AE} \left[\frac{\xi_1(1 + \alpha_1)}{3} - \xi_2(1 + \alpha_2) \right]. \quad (4.9)$$

In terms of the nondimensional parameters β_1 and β_2 , defined as

$$\beta_1 = P_{AB}/(P_E)_{AB}, \quad \beta_2 = P_{BC}/(P_E)_{BC}, \quad (4.10)$$

where $(P_E)_{AB}$ and $(P_E)_{BC}$ are the Euler buckling loads for a pinned column with the members AB and BC identical to the corresponding members in the frame, i.e., Fig. 4.1, Eq. (4.9) can be written as

$$\theta = \frac{\pi^2}{8\beta_1 L^2} k^2 \left[\frac{\xi_1(1 + \alpha_1) - 3\xi_2(1 + \alpha_2)}{\xi_1 + \sqrt{3}\xi_2} \right], \quad (4.11)$$

where k is the slenderness ratio of the members, defined as $k = (I/A)^{1/2}$, and

$$M_{BC} = \frac{2EI}{L} \xi_2 [\theta + 12(1 + \alpha_2) \beta_1 \frac{\pi^2}{L^2} k^2]. \quad (4.12)$$

It is seen from Eqs. (4.11) and (4.12) that θ depends on k and β_1 . Further, β_1 is related to the load in the member.

The critical load of a frame is the load at which the rotation θ , and hence M_{BC} , becomes very large. This condition gives the value of β_1 or β_2 from which P_{cr} , and therefore W_{cr} , of the frame can be obtained.

The interested reader may see Horne and Merchant [2] or any other standard book on structural analysis for a discussion on the moment distribution method, as also Kirby and Nethercot [3].

4.3 RIGID JOINTED FRAME

We shall first analyze a beam column member and then extend our analysis to a frame.

4.3.1 Analysis of Beam Columns

We shall adopt the equilibrium approach to study the beam columns.

(i) Figure 4.2 shows a pin jointed beam column with a moment applied at one end.

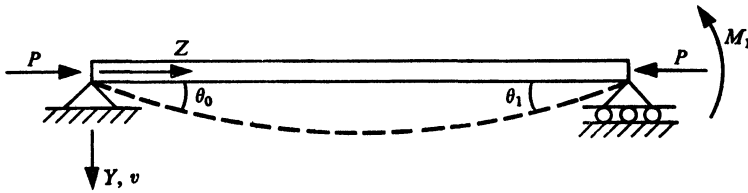


Fig. 4.2 Pin Jointed Beam Column.

The moment at any z is

$$M_x = \frac{M_1}{L}z + Pv. \quad (4.13)$$

From Eq. (1.1), we have

$$EI \frac{d^2v}{dz^2} = -M_x.$$

Substituting for M_x , we can write Eq. (1.1) as

$$\frac{d^2v}{dz^2} = -\frac{P}{EI}v - \frac{1}{EI} \frac{M_1}{L}z \quad (4.14)$$

or

$$\frac{d^2v}{dz^2} + k^2v = -\frac{M_1}{EI} \frac{z}{L}, \quad (4.15)$$

where $k^2 = P/(EI)$. It is easy to see that Eq. (4.15) is a nonhomogeneous form of Eq. (1.2). Following the steps in Section 1.2.1, we can write the solution of Eq. (4.15) as

$$v = C_1 \cos kz + C_2 \sin kz - \frac{M_1}{PL}z. \quad (4.16)$$

The coefficients C_1 and C_2 are determined from the boundary conditions which are

$$\begin{aligned} v &= 0 & (z = 0), \\ v &= 0 & (z = L). \end{aligned} \quad (4.17)$$

These give

$$C_1 = 0, \quad C_2 = \frac{M_1}{P} \frac{1}{\sin kL}.$$

Therefore,

$$v = \frac{M_1}{P} \left(\frac{\sin kz}{\sin kL} - \frac{z}{L} \right). \quad (4.18)$$

The rotations θ_0 and θ_1 at the ends of the beam column can now be obtained. Thus,

$$\begin{aligned} \theta_0 &= \frac{M_1}{P} \left(k \operatorname{cosec} kL - \frac{1}{L} \right), \\ \theta_1 &= -\frac{M_1}{P} \left(k \cot kL - \frac{1}{L} \right). \end{aligned} \quad (4.19)$$

(ii) Let us consider a little more complicated situation where, in addition to the axial load P , a transverse load W is applied. Figure 4.3 shows such a member.

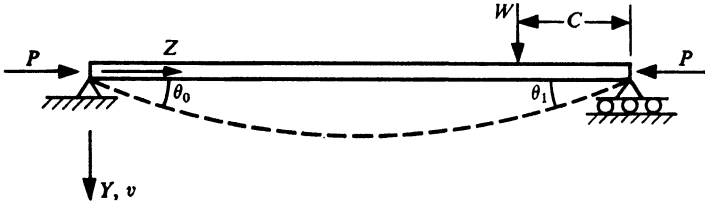


Fig. 4.3 Pin Jointed Column under Transverse Load.

The bending moment due to the applied load W , if acting alone, can be easily found. However, since P is also acting, an additional bending moment is generated. This moment can be known only after the deflections have been determined.

The bending moments at any z to the left and to the right of the load W are, respectively,

$$M_{zL} = \frac{W}{L} Cz + Pv, \quad (4.20a)$$

$$M_{zR} = \frac{W(L-C)}{L} (L-z) + Pv. \quad (4.20b)$$

Using Eq. (1.1), we obtain

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = -\frac{W}{L} \frac{C}{EI}z, \quad (4.21a)$$

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = -\frac{W}{L} \frac{(L-C)}{EI}(L-z). \quad (4.21b)$$

The solution to these can easily be written as

$$v = C_1 \cos kz + C_2 \sin kz - \frac{W}{LP}Cz, \quad (4.22a)$$

$$v = C_3 \cos kz + C_4 \sin kz - \frac{W(L-C)(L-z)}{P}. \quad (4.22b)$$

The constants C_1 , C_2 , C_3 , and C_4 are determined from the boundary conditions and the continuity conditions. The boundary conditions are

$$v = 0 \quad (z = 0),$$

$$v = 0 \quad (z = L).$$

At the point of application of the load, the deflection and the slope on either side of W must be the same. The equation for v is finally then obtained as

$$v = \frac{W \sin kC}{Pk \sin kL} \sin kz - \frac{WC}{PL}z, \quad 0 \leq z \leq (L-C), \quad (4.23a)$$

$$v = \frac{W \sin k(L-C)}{Pk \sin kL} \sin k(L-z) - \frac{W(L-C)(L-z)}{PL}, \quad (L-C) \leq z \leq L. \quad (4.23b)$$

(iii) If, in Fig. 4.2, two couples M_L and M_R are acting at the ends of the member, the deflection curve v can be obtained by superposing the two values of v . These values are gotten by substituting M_L for M_R and $(L-z)$ for z in Eq. (4.18). Combining the two values, we have

$$v = \frac{M_R}{P} \left(\frac{\sin kz}{\sin kL} - \frac{z}{L} \right) + \frac{M_L}{P} \left[\frac{\sin k(L-z)}{\sin kL} - \frac{(L-z)}{L} \right]. \quad (4.24)$$

The end moments M_L and M_R in an actual situation may appear either as applied moments or as two eccentrically applied compressive axial loads P . Substituting $M_L = Pe_L$ and $M_R = Pe_R$ in Eq. (4.24), where e_R and e_L are, respectively, the offset distances at the right and left ends, we get

$$v = e_R \left(\frac{\sin kz}{\sin kL} - \frac{z}{L} \right) + e_L \left[\frac{\sin k(L-z)}{\sin kL} - \frac{(L-z)}{L} \right]. \quad (4.25)$$

For a study of beam columns with load conditions other than those we have considered, see Timoshenko and Gere [4].

We shall now demonstrate the application of the foregoing results to simple frames.

EXAMPLE 4.1 (Buckling load of frame member) Figure 4.4 shows a frame along with the applied loads. The frame $ABCD$ is supported at the ends such that it cannot have any horizontal movement. The buckling of a frame implies the buckling of its members (here,

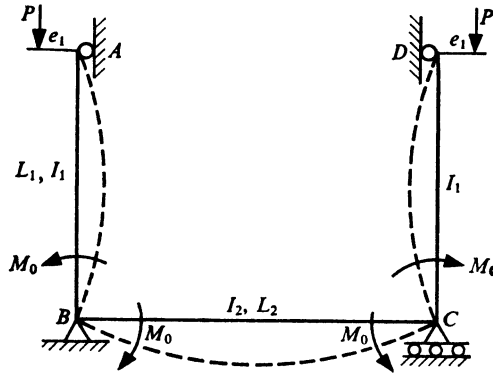


Fig. 4.4 Buckling of Frame.

AB and *CD*). Isolating the member *AB*, we find the loads and reactions on it are as in Fig. 4.5.

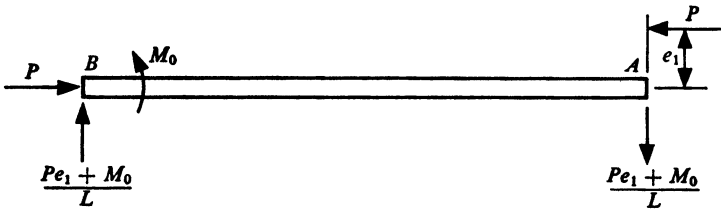


Fig. 4.5 Loads and Reactions on Member.

Following the method discussed earlier in this section, we can obtain the deflection, and hence the rotation θ_0 , at the point *B* as

$$\theta_0 = e_1(k \operatorname{cosec} kL_1 - \frac{1}{L_1}) + \frac{M_0}{P}(k \cot kL_1 - \frac{1}{L_1}). \quad (4.26)$$

Since the members are rigidly jointed at the ends *B* and *C*, for compatibility, the end *B* of the beam *BC* also has to rotate by the same amount as θ_0 . From the beam analysis (see any book on elementary structural analysis), we have

$$\theta_0 = M_0 L_2 / (2EI_2). \quad (4.27)$$

Eliminating θ_0 from Eqs. (4.26) and (4.27), we get

$$M_0 = \frac{\frac{e_1}{L_1}(kL_1 \operatorname{cosec} kL_1 - 1)}{\frac{1}{PL_1}(1 - kL_1 \cot kL_1) + \frac{L_2}{2EI_2}}. \quad (4.28)$$

For simplicity, if $L_1/I_1 = L_2/I_2$, the buckling occurs when $M_0 \rightarrow \infty$, that is, when

$$\frac{1}{(kL_1)^2}(1 - kL_1 \cot kL_1) + \frac{1}{2} = 0$$

or

$$\cot kL_1 = \frac{1}{kL_1} + \frac{kL_1}{2}. \quad (4.29)$$

Equation (4.29) is a transcendental equation, its lowest root being $kL_1 = 3.59$. Therefore,

$$P_{cr} = \frac{(kL_1)^2 EI_1}{L_1^2} = \frac{(12.9)EI_1}{L_1^2}. \quad (4.30)$$

EXAMPLE 4.2 (Buckling load of symmetric frame subjected to axial load) Consider a symmetric frame so supported that it cannot have any lateral movement (Fig. 4.6). When the axial load P reaches a critical value, the columns AB and CD will begin to deflect laterally, resulting in the bending of the beam members AD and BC . The end moments generated because of this bending restrain the members AB and CD from buckling. Thus, we treat these members as columns with elastic restraint.

Equation (4.24) gives the deflection due to the end moments M_R and M_L . Further, the rotations at the ends can be obtained by differentiating Eq. (4.24) with respect to z and evaluating the derivatives at $z = 0$ and $z = L$. This yields

$$\theta_L = \frac{M_R}{P} \left(\frac{k}{\sin kL} - \frac{1}{L} \right) + \frac{M_L}{P} \left(-\frac{k}{\tan kL} + \frac{1}{L} \right), \quad (4.31a)$$

$$\theta_R = \frac{M_R}{P} \left(-\frac{k}{\tan kL} + \frac{1}{L} \right) + \frac{M_L}{P} \left(\frac{k}{\sin kL} - \frac{1}{L} \right). \quad (4.31b)$$

Using the fact that $k^2 = P/(EI)$, we can rewrite Eqs. (4.31) as ($M_A = M_R$, $M_B = M_L$, $\theta_L = \theta_B$, $\theta_R = \theta_A$)

$$\theta_B = \frac{M_A L}{3EI} \alpha(\lambda) + \frac{M_B L}{6EI} \beta(\lambda), \quad (4.32a)$$

$$\theta_A = \frac{M_B L}{3EI} \alpha(\lambda) + \frac{M_A L}{6EI} \beta(\lambda), \quad (4.32b)$$

where

$$\beta(\lambda) = \frac{3}{\lambda} \left(\frac{1}{\sin 2\lambda} - \frac{1}{2\lambda} \right), \quad (4.33a)$$

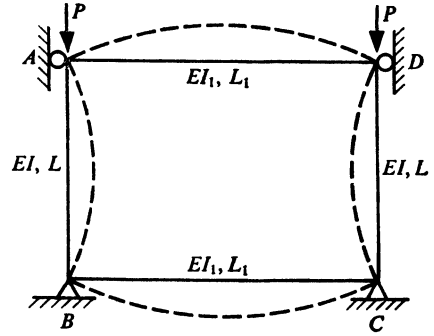


Fig. 4.6 Symmetric Frame Compressed Axially.

$$\alpha(\lambda) = \frac{3}{2\lambda} \left(-\frac{1}{\tan 2\lambda} + \frac{1}{2\lambda} \right), \quad (4.33b)$$

$$\lambda = \frac{kL}{2} = \frac{L}{2} \sqrt{\frac{P}{EI}}. \quad (4.33c)$$

Thus, the rotations implicitly contain the axial load P .

From the condition of compatibility, the rotation θ of the column must be the same as the rotation of a horizontal bar. The rotation θ of a bar is related to the bending moment M_B of the bar as

$$M_B = -\gamma\theta_B, \quad (4.34)$$

where

$$\gamma = 2EI_1/L_1. \quad (4.35)$$

Substituting for θ_B in Eq. (4.32a), we get

$$\frac{M_B}{\gamma} + \frac{M_A L}{3EI} \alpha(\lambda) + \frac{M_B L}{6EI} \beta(\lambda) = 0. \quad (4.36)$$

For the frame in Fig. 4.6, $M_A = M_B$. Hence,

$$\frac{1}{\gamma} + \frac{L}{3EI} \alpha(\lambda) + \frac{L}{6EI} \beta(\lambda) = 0. \quad (4.37)$$

Substituting for γ , $\alpha(\lambda)$, and $\beta(\lambda)$ in Eq. (4.37), we find this equation reduces to the transcendental equation for critical load P given by

$$\frac{\tan \lambda}{\lambda} = -\frac{L_1}{I_1} \frac{I}{L}. \quad (4.38)$$

If all the four members of the frame are identical, Eq. (4.38) becomes

$$\tan \lambda = -\lambda. \quad (4.39)$$

The lowest root of this transcendental equation gives the critical load as

$$P_{cr} = (16.47) \frac{EI}{L^2}. \quad (4.40)$$

EXAMPLE 4.3 (Buckling load of portal frame with sway) Consider a frame subjected to an axial load P (Fig. 4.7). As P reaches P_{cr} , the column members deflect and, if the beam BC is not infinitely rigid, the joints B and C undergo a lateral motion. The dashed lines in Fig. 4.7 show the deflected shape of the frame.

Isolating a vertical member, we can write the differential equation governing the deflection as

$$EI \frac{d^2 v}{dz^2} = P(\delta - v) - M \quad (4.41)$$

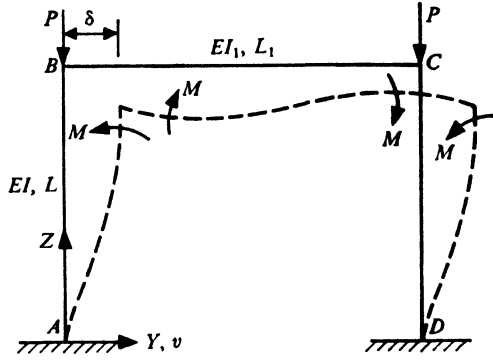


Fig. 4.7 Portal Frame under Axial Load.

or

$$EI \frac{d^2 v}{dz^2} + Pv = P\delta - M. \quad (4.42)$$

This is a nonhomogeneous ordinary differential equation with constant coefficients. Its solution is

$$v = C_1 \sin kz + C_2 \cos kz + \delta - \frac{M}{P}, \quad (4.43)$$

where $k^2 = P/(EI)$. The constants C_1 and C_2 are obtained from the support conditions at A , which are

$$v = 0, \quad dv/dz = 0 \quad (z = 0). \quad (4.44)$$

We get the deflection curve as

$$v = \left(\delta - \frac{M}{P}\right)(1 - \cos kz). \quad (4.45)$$

Equation (4.45) still has unknowns, i.e., δ and M . To evaluate these unknowns, we can set up two more conditions, namely,

$$\begin{aligned} v &= \delta & (z = L), \\ \frac{dv}{dz} &= \frac{ML_1}{6EI_1} & (z = L). \end{aligned} \quad (4.46)$$

These lead to

$$\begin{aligned} \delta \cos kL + \frac{M}{P}(1 - \cos kL) &= 0, \\ \delta k \sin kL - \frac{M}{P}(k \sin kL + \frac{L_1 P}{6EI_1}) &= 0. \end{aligned} \quad (4.47)$$

Equations (4.47) are a set of homogeneous algebraic equations and, for a nontrivial solution, the determinant of the coefficients of δ and M/P is zero. This gives the transcendental equation

$$\frac{kL}{\tan kL} = -\frac{6LI_1}{L_1I}. \quad (4.48)$$

Knowing the geometry of the frame, we can obtain P_{cr} .

4.3.2 Method of Moment Equations

The moment equations can be used to determine the stability condition when two or more members in a frame meet at a point. The method using such equations we shall give here is based on the work by Bleich [1].

Consider two bars AB and BC meeting at the rigid joint r , as shown in Fig. 4.8. Let L_k and L_{k+1} be the lengths of these members and let the members be acted upon by the compressive forces P_k and P_{k+1} (see Fig. 4.8). The figure also shows the original position of the bars and the displaced position of the bars due to buckling.

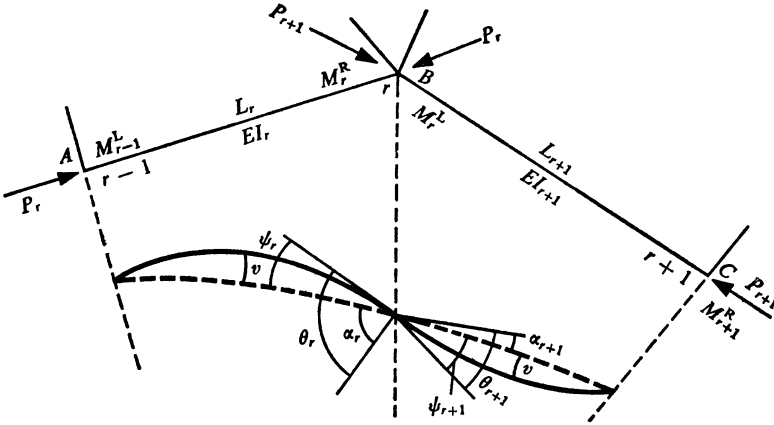


Fig. 4.8 Original and Deflected Shapes of Bars under Axial Loads.

The moments at the ends of the bars are M_{r-1}^L , M_r^R , M_r^L , and M_{r+1}^R . Since the joint at B is rigid, for continuity, the angle $\theta_r = \alpha_r + \psi_r$ must be equal to $\theta_{r+1} = \alpha_{r+1} + \psi_{r+1}$. Hence,

$$\alpha_r + \psi_r = \alpha_{r+1} + \psi_{r+1}. \quad (4.49)$$

Here, α_r and α_{r+1} are the rigid bar rotations, and ψ_r and ψ_{r+1} are the slopes of the bars given by

$$\psi_r = \left(\frac{dv_r}{dz}\right)_{z=l}, \quad \psi_{r+1} = \left(\frac{dv_{r+1}}{dz}\right)_{z=0}, \quad (4.50)$$

$$\psi_r = \frac{M_{r-1}^L}{P_r L_r} \left(-\frac{\phi_r}{\sin \phi_r} + 1 \right) + \frac{M_r^R}{P_r} (\phi_r \cot \phi_r - 1), \quad (4.51a)$$

$$\psi_{r+1} = \frac{M_r^L}{P_{r+1} L_{r+1}} (-\phi_{r+1} \cot \phi_{r+1} + 1) + \frac{M_{r+1}^R}{P_{r+1} L_{r+1}} \left(\frac{\phi_{r+1}}{\sin \phi_{r+1}} - 1 \right), \quad (4.51b)$$

where

$$\phi_r = L_r \sqrt{\frac{P_r}{EI}}, \quad \phi_{r+1} = L_{r+1} \sqrt{\frac{P_{r+1}}{EI}}.$$

Equations (4.51) are obtained by differentiating Eq. (4.24) with respect to z and substituting the corresponding values for M_R and M_L .

In terms of the notations

$$L'_r = \frac{I}{I_r} L_r, \quad L'_{r+1} = \frac{I}{I_{r+1}} L_{r+1}, \quad (4.52a)$$

we have

$$P_r L_r = \frac{EI}{L'_r} \phi_r^2, \quad P_{r+1} L_{r+1} = \frac{EI}{L'_{r+1}} \phi_{r+1}^2. \quad (4.52b)$$

In view of relations (4.52) and with the notation

$$s = \frac{1}{\phi^2} \left(\frac{\phi}{\sin \phi} - 1 \right), \quad C = \frac{1}{\phi^2} (1 - \phi \cot \phi), \quad (4.53)$$

Eqs. (4.51) take the form

$$\begin{aligned} \psi_r &= -\frac{1}{EI} (M_{r-1}^L L'_r s_r + M_r^R L'_r C_r), \\ \psi_{r+1} &= \frac{1}{EI} (M_r^R L'_{r+1} C_{r+1} + M_{r+1}^L L'_{r+1} s_{r+1}). \end{aligned} \quad (4.54)$$

Substituting the expressions for ψ_r and ψ_{r+1} in relation (4.49), we get

$$M_{r-1}^L L'_r s_r + M_r^R L'_r C_r + M_r^L L'_{r+1} C_{r+1} + M_{r+1}^R L'_{r+1} s_{r+1} - EI(\alpha_r - \alpha_{r+1}) = 0. \quad (4.55)$$

Equation (4.55) represents the relation between the end moments acting on two adjacent, rigidly connected members. It is called the *four-moment equation*.

If only two members are meeting at the r -th joint, M_r^R then becomes equal to M_r^L and Eq. (4.55) simplifies to the well-known three-moment equation

$$M_{r-1}^L L'_r s_r + M_r (L'_r C_r + L'_{r+1} C_{r+1}) + M_{r+1}^L L'_{r+1} s_{r+1} - EI(\alpha_r - \alpha_{r+1}) = 0. \quad (4.56)$$

We shall now demonstrate the application of Eq. (4.56) to obtain the buckling load of a frame through an example.

EXAMPLE 4.4 (Buckling load of portal frame hinged at ends) Consider a portal frame

subjected only to axial loads (Fig. 4.9). In view of the symmetry of the structure and the loading, we can carry out the analysis for two modes, namely, (i) symmetric and (ii) anti-symmetric.

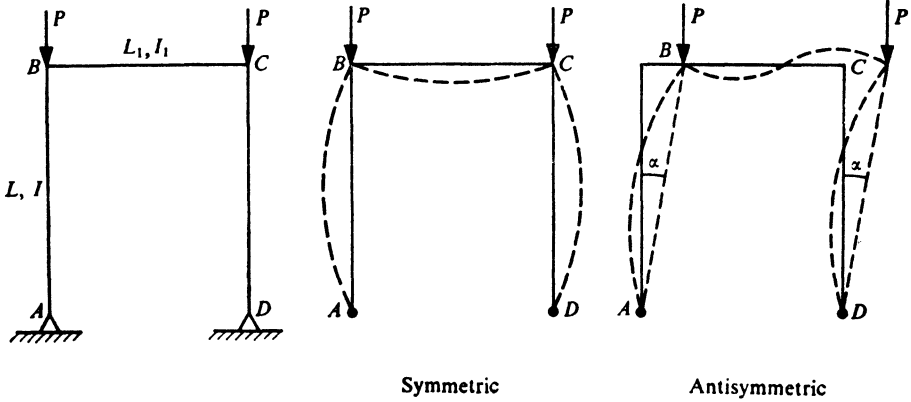


Fig. 4.9 Portal Frame with Symmetric and Antisymmetric Modes.

(i) *Symmetric mode* Because of the symmetry of the structure and the loading, the moments at the joints B and C are equal. Further, as the ends A and D are hinged, the support moments are zero. In this mode, the joints B and C do not move, but they do rotate (see Fig. 4.9).

Equation (4.56), when applied to the members AB and BC , results in

$$M_1(C_1 L' + C_2 L_1) + M_1 s_2 L_1 = 0, \quad (4.57)$$

where C_1 refers to a vertical member and C_2, s_2 refer to the horizontal member. If we apply Eq. (4.56) to BC and CD , the resulting equation is the same as Eq. (4.57) since the structure is symmetric. Further, as $M_1 \neq 0$, we get

$$C_1 + \frac{L'_1}{L'}(C_2 + s_2) = 0. \quad (4.58)$$

From Eqs. (4.52), we have

$$L' = L, \quad L'_1 = \frac{I}{I_1} L_1. \quad (4.59)$$

Now, Eq. (4.58) reduces to

$$C_1 + \frac{I}{I_1} \frac{L_1}{L}(C_2 + s_2) = 0. \quad (4.60)$$

Since there is no axial load acting on the member BC , $\phi_2 = 0$ in view of Eq. (4.52b). Further,

$s_2 = 1/6$ and $C_2 = 1/3$ from Eq. (4.52a). Hence,

$$C_1 + \frac{1}{2} \frac{I}{I_1} \frac{L_1}{L} = 0. \quad (4.61)$$

Substituting for C_1 from relations (4.53), we get

$$\frac{1}{\phi_1^2} (1 - \phi_1 \cot \phi_1) + \frac{1}{2} \frac{I}{I_1} \frac{L_1}{L} = 0. \quad (4.62)$$

Knowing the geometry of the members, we can obtain P_{cr} from the transcendental equation (4.62).

(ii) *Antisymmetric mode* In this mode, the vertical member AB undergoes rotation, but the horizontal member does not. Further, the moments at B and C are M_1 and $-M_1$, respectively.

The moment equation (4.56) now reduces to

$$M_1(L'C_1 + L'_1C_2) - M_1L_1s_2 - EI\alpha_1 = 0. \quad (4.63)$$

Isolating the member AB and applying the equilibrium equation (4.56), we get

$$M_1 = PL\alpha_1. \quad (4.64)$$

Substituting for α_1 from Eq. (4.64) in Eq. (4.63), we have

$$M_1[PL(L'C_1 + L'_1C_2 - L_1s_2) - EI] = 0. \quad (4.65)$$

Since $M_1 \neq 0$, for stability, we get

$$PL(L'C_1 + L'_1C_2 - L_1s_2) - EI = 0. \quad (4.66)$$

In view of Eqs. (4.59), Eq. (4.66) reduces to

$$PL[LC_1 + \frac{I}{I_1}L'_1(C_2 - s_2)] - EI = 0.$$

Here, again, $\phi_2 = 0$ and $s_2 = 1/6$, $C_2 = 1/3$. Hence,

$$C_1 + \frac{1}{6} \frac{I}{I_1} \frac{L_1}{L} - \phi_1^2 = 0. \quad (4.67)$$

Substituting for C_1 from Eqs. (4.53), we find Eq. (4.67) reduces to

$$\frac{\tan \phi_1}{\phi_1} - \frac{1}{6} \frac{I}{I_1} \frac{L_1}{L} = 0. \quad (4.68)$$

We can now obtain the buckling load from this equation.

4.3.3 Geometrical Approach (Haarman's Method)

In a frame structure made up of straight members, a clear picture of the physical phenomenon can be had by using the geometrical approach suggested by Haarman. However, it was Bijlaard [5] who first brought to focus the idea of this method. In what follows, we shall discuss the technique in a limited sense. For more details on it, the interested reader may see McGuire [6].

In the Haarman method, the prerequisite is the knowledge of the buckled configuration of the axially loaded bar being analyzed with any boundary conditions. Further, the coordinate system is established, taking the origin at one flex point and selecting the axis passing through the other flex point. This results in a simple sine curve description for the deflection. By satisfying the given boundary or compatibility conditions, the buckling load can be found.

We shall illustrate the method first through some simple elements and then apply it to complicated problems.

EXAMPLE 4.5 (Buckling of column hinged at top and clamped at base) Consider the column shown in Fig. 4.10. The point C here represents one flex point and B the other flex point. Let us choose the point B as the origin and the axis passing through B and C as the Z -axis. Further, assume that the Y -axis is normal to the Z -axis. The Z -axis makes an infinitesimally small angle θ with the vertical. The offset δ is the intersection of the Z -axis with the base.

With the increase in P , the column starts deflecting and, at buckling, the resultant force P_{cr} becomes inclined since the axis of P must pass through the two flex points. This results in the development of a horizontal reaction H and a vertical load P'_{cr} . However, $P_{cr} \approx P'_c$ because the displacement is infinitesimal.

The equation for the displacement curve between the points B and C is written as

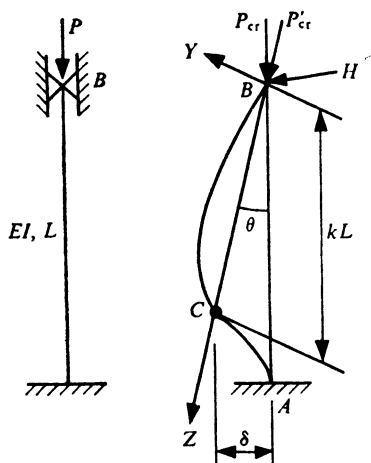


Fig. 4.10 Column and Its Deflected Shape.

$$v = v_0 \sin \frac{\pi z}{kL}, \quad (4.69)$$

where v is the displacement along the Y -axis, v_0 is the maximum displacement, and $k = \sqrt{P/(EI)}$. Hence,

$$\frac{dv}{dz} = v_0 \frac{\pi}{kL} \cos \frac{\pi z}{kL}. \quad (4.70)$$

Equation (4.69) has two unknowns, namely, v_0 and k , and requires two conditions to evaluate them. These conditions are

$$\begin{aligned} v &= -\delta & (z = L), \\ dv/dz &= -\alpha & (z = L). \end{aligned} \quad (4.71)$$

Further, $\alpha = \delta/L$. Applying now the boundary conditions (4.71), we get

$$\tan \frac{\pi}{k} = \frac{\pi}{k}. \quad (4.72)$$

From this, we can obtain the critical load.

EXAMPLE 4.6 (Buckling of pin jointed portal frame) Consider, once again, the portal

frame in Example 4.4. As in that example, so too here we can carry out the analysis in two ways, namely, when there is (i) no side sway and (ii) side sway.

(i) *Without side sway (symmetric mode)* Before buckling, all members are straight, but as the applied axial load reaches the critical value, the frame assumes the configuration

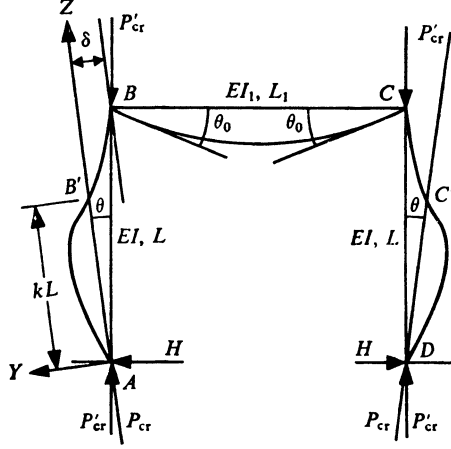


Fig. 4.11 Portal Frame under Axial Load (Symmetric Mode).

shown in Fig. 4.11. Here, B' and C' represent the flex points. The equation for the curve AB' is

$$v = v_0 \sin \frac{\pi z}{kL}, \quad (4.73)$$

$$\frac{dv}{dz} = \frac{\pi}{kL} v_0 \cos \frac{\pi z}{kL}. \quad (4.74)$$

At $z = L$,

$$v = -\delta = -v_0 \sin \frac{\pi}{k}, \quad (4.75)$$

$$\theta = \frac{\delta}{L} = -\frac{v_0}{L} \sin \frac{\pi}{k}. \quad (4.76)$$

From Fig. 4.11,

$$\theta + \theta_0 = -\left(\frac{dv}{dz}\right)_{z=L}$$

or

$$\theta_0 = -\theta - \frac{\pi}{kL} v_0 \cos \frac{\pi}{k}. \quad (4.77)$$

Following Eq. (4.27), we have, for a beam with two end moments,

$$\theta_0 = \frac{M_0 L_1}{2EI_1}, \quad M_0 = P_{cr} \delta.$$

Therefore,

$$\theta_0 = \frac{P_{cr} \delta L_1}{2EI_1}. \quad (4.78)$$

From Eq. (4.76), we then have

$$-P_{cr} \frac{v_0}{L} L_1 \frac{\sin(\pi/k)}{2EI_1} = \frac{v_0}{L} \sin \frac{\pi}{k} - \frac{\pi}{kL} v_0 \cos \frac{\pi}{k}. \quad (4.79)$$

Since

$$P_{cr} = \frac{\pi^2 EI}{(kL)^2},$$

Eq. (4.79) reduces to

$$\frac{\pi}{k} \cot \frac{\pi}{k} = 1 + \frac{1}{2} \frac{\pi^2}{k^2} \frac{I}{L} \frac{L_1}{I_1}. \quad (4.80)$$

Equation (4.80) gives the relation for P_{cr} .

(ii) *With side sway (antisymmetric mode)* If the structure and the loading are symmetric, there is no bending or lateral motion prior to collapse. At buckling, lateral motion is possible, resulting in an antisymmetric buckled configuration as shown in Fig. 4.12. Now,

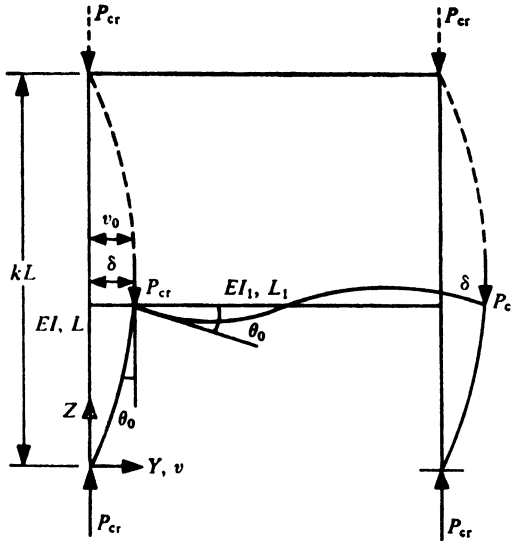


Fig. 4.12 Portal Frame under Axial Load (Antisymmetric Mode).

as before,

$$v = v_0 \sin \frac{\pi z}{kL}. \quad (4.81)$$

Hence,

$$\theta_0 = \left(\frac{dv}{dz} \right)_{z=L} = \frac{\pi v_0}{kL} \cos \frac{\pi}{k}. \quad (4.82)$$

For the beam,

$$\theta_0 = \frac{M_0 L_1}{2EI_1} = \frac{P_{cr} \delta L_1}{6EI_1}. \quad (4.83)$$

Equating Eqs. (4.82) and (4.83), after substituting for P_{cr} and δ , we get

$$\cot \frac{\pi}{k} = \frac{1}{6} \frac{\pi}{k} \frac{I}{L} \frac{L_1}{I_1}. \quad (4.84)$$

Equation (4.84) gives the buckling load.

4.3.4 Frames with Primary Bending

In Examples 4.5 and 4.6, we studied the buckling of frames with no applied loads on the horizontal members (i.e., beams). In a similar manner, we can examine the buckling characteristics of frames with external loads on the beams. The frames can be with or without sway.

EXAMPLE 4.7 (Buckling of pin jointed portal frame with transverse loads on beam)
Consider, yet again, the portal frame in Example 4.4. Assume that it is loaded as in Fig. 4.13.

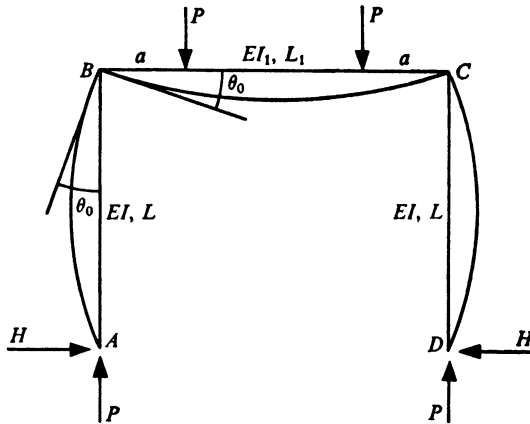


Fig. 4.13 Portal Frame under Transverse Loads.

As can be seen, the external loads acting on the beam are equal and symmetrically placed. Further, at A and D , the reactions are P and H .

We shall use the method of beam columns (discussed in Section 4.3.1) to obtain the buckling characteristics. Isolating the vertical member AB and indicating all the loads on it, we find that the situation is identical to case (i) (Section 4.3.1). The moment M_1 is equivalent to HL . The slope of the column at B is given by the second of Eqs. (4.19), that is,

$$\theta_0 = -\frac{HL}{P}(k \cot kL - \frac{1}{L}),$$

where $k = \sqrt{P/(EI)}$. Letting $kL = \pi/m$, we get

$$\theta_0 = -\frac{H}{P}(\frac{\pi}{m} \cot \frac{\pi}{m} - 1). \quad (4.85)$$

The slopes of the beam at B and C are obtained by superimposing the solutions obtained in Section 4.3.1 for cases (i), (ii), and (iii). This yields

$$\begin{aligned} \theta_0 = & -\frac{HL}{H}(k_2 \operatorname{cosec} k_2 L_1 - \frac{1}{L_1}) + \frac{HL}{H}(k_2 \cot q_2 L_1 - \frac{1}{L_1}) \\ & + \frac{P}{H}\{(\sec \frac{k_2 L_1}{2})[\cos k_2 L_1(\frac{1}{2} - \frac{d}{L_1})] - 1\}, \end{aligned} \quad (4.86)$$

where

$$k_2 = \sqrt{H/(EI_1)}, \quad k_2 L_1 = \pi/n.$$

Eliminating θ_0 from Eqs. (4.85) and (4.86), the equation governing the buckling load we get is

$$\frac{\pi L}{n L_1} \tan \frac{\pi}{2n} + \frac{H}{P}(1 - \frac{\pi}{m} \cot \frac{\pi}{m}) - \frac{P}{H}[\sec \frac{\pi}{2n} \cos \frac{\pi}{n}(\frac{1}{2} - \frac{d}{L_1}) - 1] = 0, \quad (4.87)$$

where

$$\frac{\pi}{m} = L \sqrt{\frac{P}{EI}}, \quad \frac{\pi}{n} = L_1 \sqrt{\frac{H}{EI_1}}.$$

Equation (4.87) is the general equation relating the horizontal force H to the applied transverse load P . As P increases, H varies nonlinearly. Further, it is a highly complex equation, and hence it is not possible to indicate, in general, the values of P for which the frame will buckle. In a given situation, if $I = I_1$, $L = L_1$, and $d = L_1/3$, Eq. (4.87) reduces to

$$\frac{\pi}{n} \tan \frac{\pi}{2n} + \frac{m^2}{n^2}(1 - \frac{\pi}{m} \cot \frac{\pi}{m}) - \frac{n^2}{m^2}[\frac{1}{6} \sec \frac{\pi}{2n} \cos \frac{\pi}{n} - 1] = 0 \quad (4.88)$$

with

$$\frac{H}{P} = \frac{m^2}{n^2}.$$

Figure 4.14 shows a plot of the nondimensional load P versus the horizontal reaction H . It can be observed that as P increases, H also increases initially. Subsequently, H begins to decrease, ultimately becoming zero at $P_{cr} = \pi^2 EI/L^2$. This behaviour of H can be explained as follows. As P increases, the capacity of the column to sustain the moment decreases, and as such, H must also decrease. Finally, a stage is reached when the column cannot resist any moment and the column moment (HL), and hence H , becomes zero.

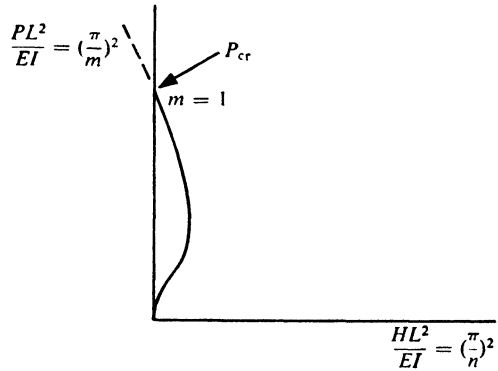


Fig. 4.14 Variation of P with H .

4.3.5 Buckling of Multistoreyed-Multibay Frames

The instances we have so far considered are not of much practical interest. In real world, multistoreyed frames are more common. Further, it is clear from our foregoing discussion that, mathematically, the analysis is quite involved. Thus, for a two-storeyed frame, the method of moment equations (Section 4.3.2) can be used. However, for a frame having more than two storeys, this method becomes quite cumbersome. In such a situation, it is convenient to apply Haarman's method (Section 4.3.3).

EXAMPLE 4.8 (Buckling of two-storeyed frame clamped at base) Figure 4.15 depicts a two-storeyed rectangular frame loaded as shown. In our analysis, we shall consider the two possible deflection modes, namely, (i) symmetric and (ii) unsymmetric.

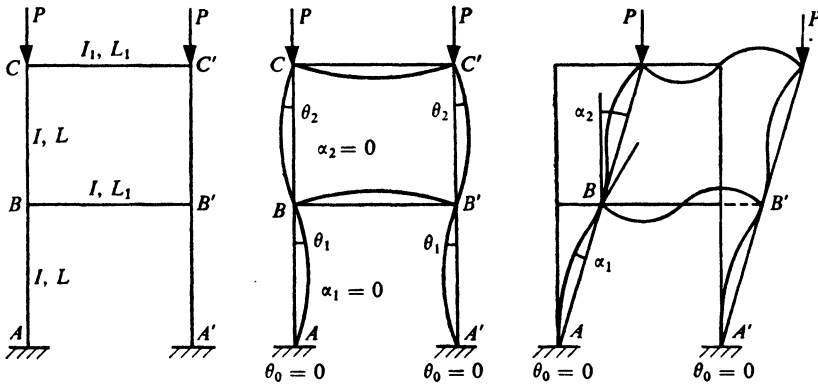


Fig. 4.15 Two-Storeyed Frame with Deformed Shapes.

Equations (4.54) can be rewritten, after manipulation, for the beam element subjected to the moments M_A and M_B (see Fig. 4.16), as

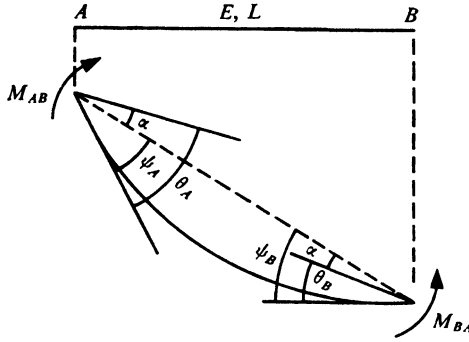


Fig. 4.16 Beam Segment with End Moments.

$$\begin{aligned}
 M_{AB} &= \frac{EI}{L'} \left[\frac{C}{C^2 - s^2} \theta_A + \frac{s}{C^2 - s^2} \theta_B - \alpha \left(\frac{C}{C^2 - s^2} + \frac{s}{C^2 - s^2} \right) \right], \\
 M_{BA} &= -\frac{EI}{L'} \left[\frac{s}{C^2 - s^2} \theta_A + \frac{C}{C^2 - s^2} \theta_B - \alpha \left(\frac{C}{C^2 - s^2} + \frac{s}{C^2 - s^2} \right) \right],
 \end{aligned} \tag{4.89}$$

where

$$L' = \frac{I}{I_{AB}} L.$$

(i) *Symmetric mode* For this case, $\alpha_1 = 0$ and $\alpha_2 = 0$. Applying Eqs. (4.89) to the vertical members AB and BC , we get

$$M_{AB} = \frac{EI}{L} \frac{s}{C^2 - s^2} \theta_1 \tag{4.90a}$$

(for member AB),

$$M_{BA} = -\frac{EI}{L} \frac{C}{C^2 - s^2} \theta_1 \tag{4.90b}$$

$$M_{BC} = \frac{EI}{L} \left(\frac{C}{C^2 - s^2} \theta_1 + \frac{s}{C^2 - s^2} \theta_2 \right) \tag{4.91a}$$

(for member BC).

$$M_{CB} = -\frac{EI}{L} \left(\frac{s}{C^2 - s^2} \theta_1 + \frac{C}{C^2 - s^2} \theta_2 \right) \tag{4.91b}$$

Further, for the horizontal members BB' and CC' , we have

$$M_{BB'} = \frac{EI_1}{L_1} \left(\frac{C_1}{C_1^2 - s_1^2} \theta_1 - \frac{s_1}{C_1^2 - s_1^2} \theta_2 \right), \tag{4.92a}$$

$$M_{CC'} = -\frac{EI_1}{L_1} \left(\frac{C_1}{C_1^2 - s_1^2} \theta_2 - \frac{s_1}{C_1^2 - s_1^2} \theta_1 \right). \tag{4.92b}$$

The equilibrium equations at the joints B and C are

$$M_{BA} - M_{BC} - M_{BB'} = 0, \quad (4.93a)$$

$$M_{CB} - M_{CC'} = 0. \quad (4.93b)$$

Substituting the moment relations in Eqs. (4.93), we obtain

$$\theta_1 \left(\frac{2C}{C^2 - s^2} + \beta \frac{C_1 - s_1}{C_1^2 - s_1^2} \right) + \theta_2 \frac{s}{C^2 - s^2} = 0, \quad (4.94a)$$

$$\theta_1 \frac{s}{C^2 - s^2} + \theta_2 \left(\frac{C}{C^2 - s^2} + \beta \frac{C_1 - s_1}{C_1^2 - s_1^2} \right) = 0. \quad (4.94b)$$

For a nontrivial solution of θ_1 and θ_2 , the determinant of the coefficients of θ_1 and θ_2 must be zero. This yields the stability equation

$$(2C' + \gamma)(C' + \gamma) - S^2 = 0, \quad (4.95)$$

where

$$C' = \frac{C}{C^2 - s^2}, \quad S = \frac{s}{C^2 - s^2}, \quad \gamma = \frac{1}{C_1 + s_1}.$$

We can now obtain the buckling load P from Eq. (4.95).

(ii) *Antisymmetric mode* The moment equations for the vertical members AB and BC here are

$$M_{AB} = \frac{EI}{L} [S\theta_1 - \alpha_1(C + S)] \quad (4.96a)$$

(for member AB),

$$M_{BA} = -\frac{EI}{L} [C\theta_1 - \alpha_1(C + S)] \quad (4.96b)$$

$$M_{BC} = \frac{EI}{L} [\theta_1 + S\theta_2 - \alpha_2(C + S)] \quad (4.97a)$$

(for member BC).

$$M_{CB} = -\frac{EI}{L} [S\theta_1 + C\theta_2 - \alpha_2(C + S)] \quad (4.97b)$$

Further, for the horizontal members BB' and CC' , we have

$$M_{BB'} = \frac{EI_1}{L_1} (C_1\theta_1 + S_1\theta_1), \quad (4.98a)$$

$$M_{CC'} = \frac{EI_1}{L_1} (C_1\theta_2 + S_1\theta_2). \quad (4.98b)$$

In addition to the equilibrium equations (4.93), we have the following equations for the equilibrium of the vertical members AB and BC :

$$M_{BA} = M_{AB} + PL\alpha_1, \quad (4.99a)$$

$$M_{CB} = M_{BC} + PL\alpha_2. \quad (4.99b)$$

Substituting the moment relations in Eqs. (4.93) and (4.99), we get a set of four simultaneous algebraic equations in θ_1 , θ_2 , α_1 , and α_2 . For a nontrivial solution, the determinant of the coefficients of these must be zero. This yields the characteristic equation for obtaining the buckling load.

EXAMPLE 4.9 (Buckling of multistoreyed frame with no side sway) Figure 4.17 shows a multistoreyed frame. In it, all the vertical members have the same area of cross-section. However, the second moment of area of the horizontal members varies from one storey to another. Further, the vertical members are subjected to axial loads as shown. Moreover, the frame does not undergo any horizontal movement.

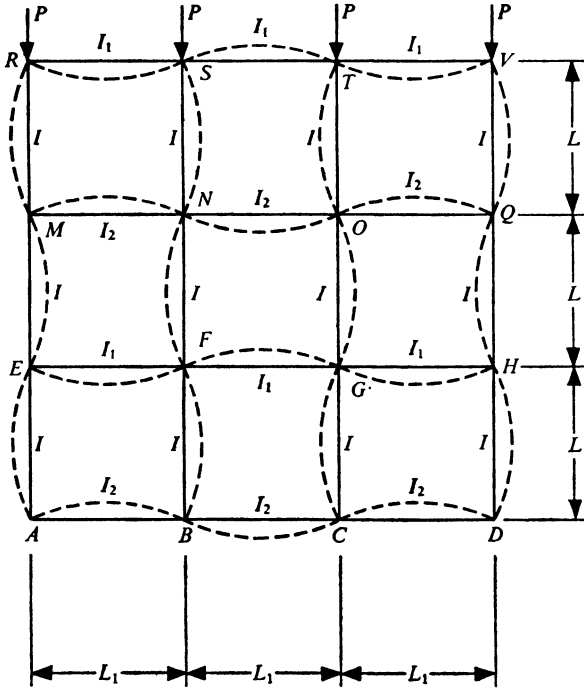


Fig. 4.17 Multistoreyed Frame with Deflected Shape.

As P approaches P_{cr} , all the columns buckle simultaneously as shown in Fig. 4.17. Since the structure is symmetric, equal moments exist at each beam end. Figure 4.18 shows the exploded view of the column FN .

In our analysis here, we shall use the technique we discussed in Section 4.3.3. As already noted, the equation for the column between two flex points is

$$v = v_0 \sin \frac{\pi z}{kL}. \quad (4.100)$$

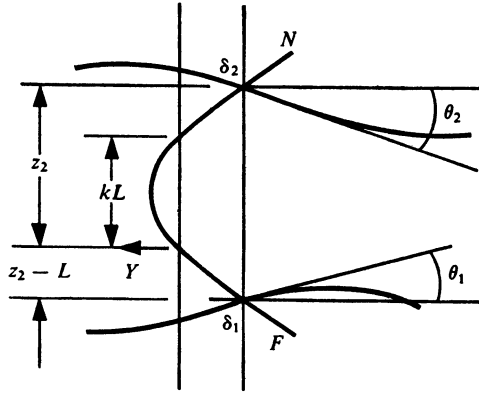


Fig. 4.18 Exploded View of Column.

Hence, the offsets

$$\begin{aligned}\delta_1 &= -v_0 \sin \frac{\pi z_2}{kL}, \\ \delta_2 &= -v_0 \left(\sin \frac{\pi z_2}{kL} \cos \frac{\pi}{k} - \cos \frac{\pi z_2}{kL} \sin \frac{\pi}{k} \right).\end{aligned}\tag{4.101a}$$

For a small deformation, $\alpha = (\delta_2 - \delta_1)/L$. Hence,

$$\alpha = -\frac{v_0}{L} \left[\left(1 - \cos \frac{\pi}{k} \right) \sin \frac{\pi z_2}{kL} + \cos \frac{\pi z_2}{kL} \sin \frac{\pi}{k} \right].\tag{4.101b}$$

The slopes of the beam at the joints N and F are θ_2 and θ_1 , respectively, and are given by

$$\theta_1 = -\left. \frac{\partial v}{\partial z} \right|_{z=z_1} - \alpha, \quad \theta_2 = \left. \frac{dv}{dz} \right|_{z=z_1-L} + \alpha.\tag{4.101c}$$

Substituting for $\partial v/\partial z$ and α from Eqs. (4.100) and (4.101b), we get

$$\theta_1 = \frac{v_0}{L} \left[\left(1 - \cos \frac{\pi}{k} \right) \sin \frac{\pi z_2}{kL} + \left(\sin \frac{\pi}{k} - \frac{\pi}{k} \right) \cos \frac{\pi z_2}{kL} \right],\tag{4.102a}$$

$$\theta_2 = \frac{v_0}{L} \left[\left(\frac{\pi}{k} \sin \frac{\pi}{k} - 1 + \cos \frac{\pi}{k} \right) \sin \frac{\pi z_2}{kL} + \left(\frac{\pi}{k} \cos \frac{\pi}{k} - \sin \frac{\pi}{k} \right) \cos \frac{\pi z_2}{kL} \right].\tag{4.102b}$$

For the beam, we have, from Eq. (4.78),

$$\theta_2 = \frac{P_{cr} \delta_2 L_1}{2EI_1}.\tag{4.103}$$

Defining

$$P_{cr} = \frac{\pi^2 EI}{k^2 L^2}$$

and substituting for δ_2 in Eq. (4.103), we get

$$\theta_2 = -\frac{v_0}{2L} \left(\frac{\pi}{k}\right)^2 \frac{IL_1}{I_2 L} \sin \frac{\pi z_2}{kL} \quad (4.104a)$$

Similarly,

$$\theta_1 = -\frac{v_0}{2L} \left(\frac{\pi}{k}\right)^2 \frac{IL_1}{I_1 L} \left(\sin \frac{\pi z_2}{kL} \cos \frac{\pi}{k} - \cos \frac{\pi z_2}{kL} \sin \frac{\pi}{k} \right). \quad (4.104b)$$

Eliminating θ_1 and θ_2 from Eqs. (4.102a), (4.102b), (4.104a), and (4.104b), the characteristic equation for the buckling load we get is

$$\frac{1}{4} \frac{I^2 L_1^2}{I_1 I_2 L^2} \left(\frac{\pi}{k}\right)^2 + \frac{I L_1}{L} \left(\frac{I_1 + I_2}{I_1 I_2}\right) \left(1 - \frac{\pi}{k \tan \frac{\pi}{k}}\right) + \frac{2k \tan \frac{\pi}{2k}}{\pi} = 1. \quad (4.105)$$

We can now evaluate k if we know the characteristic dimensions of the frame.

PROBLEMS

4.1 Find the buckling load for the two-bar structure shown in Fig. 4.19 when:

- (i) I_2 is small as compared with I_1 .
- (ii) $I_1 = \sqrt{3}I_2$.
- (iii) I_1 is small as compared with I_2 .

The joint B is a rigid joint.

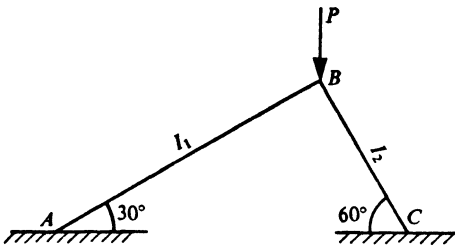


Fig. 4.19 Problem 4.1.

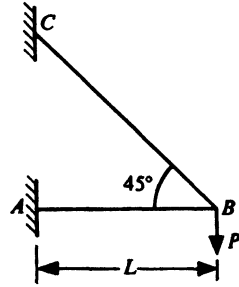


Fig. 4.20 Problem 4.2.

4.2 For the cantilever bracket shown in Fig. 4.20, obtain the critical value of P if A and C are clamped and the joint B is rigid. Assume that the members are of uniform cross-section.

4.3 Determine the critical load for the member BC of the frame shown in Fig. 4.21. The joints B and C are rigid.

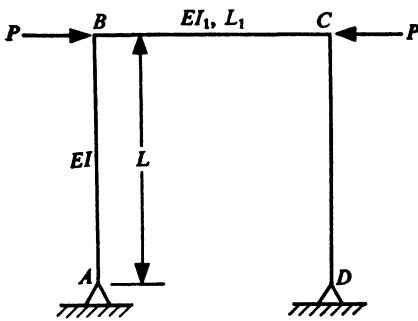


Fig. 4.21 Problem 4.3.

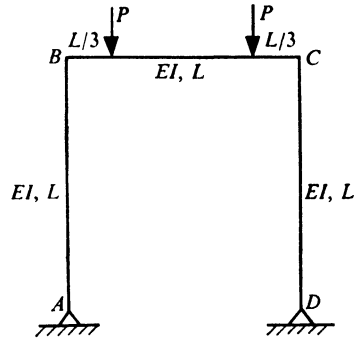


Fig. 4.22 Problem 4.4.

4.4 Determine the critical load of a frame (see Fig. 4.22) subjected to transverse loading.

4.5 What would be the critical load if the loads in Fig. 4.21 are shifted to the points B and C ?

4.6 Obtain the buckling load of the member AB of the two-storeyed frame shown in Fig. 4.23 if the frame has

- (i) no side sway,
- (ii) side sway.

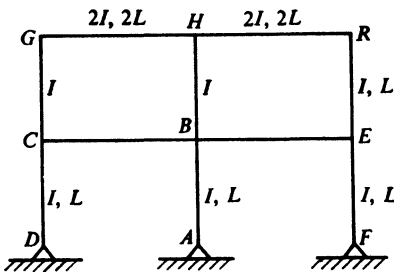


Fig. 4.23 Problem 4.6.

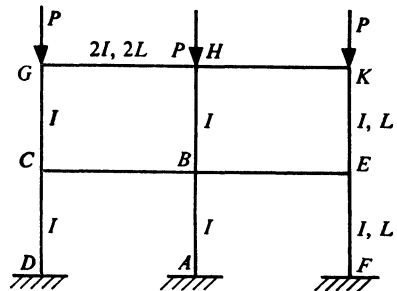


Fig. 4.24 Problem 4.7.

4.7 Do the same as in Problem 4.6 when the ends D , A , and F are clamped to the base (see Fig. 4.24).

4.8 Estimate the critical load of a braced portal frame (Fig. 4.25) with no sway.

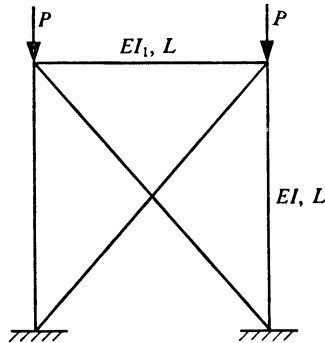


Fig. 4.25 Problem 4.8.

4.9 A 2.5-metre-long steel column is clamped at the bottom and rigidly attached to a 1.25-metre-long horizontal steel beam at the top, as shown in Fig. 4.26. Determine the critical load for elastic buckling in the plane of the frame. The bending stiffness EI for the column is 2305 GPa and that for the beam is 588.6 GPa.

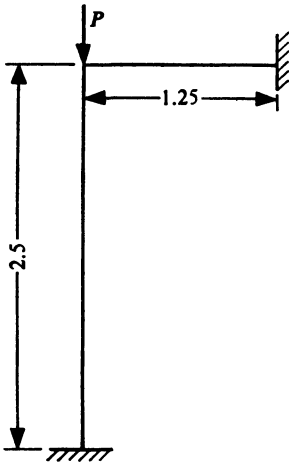


Fig. 4.26 Problem 4.9.

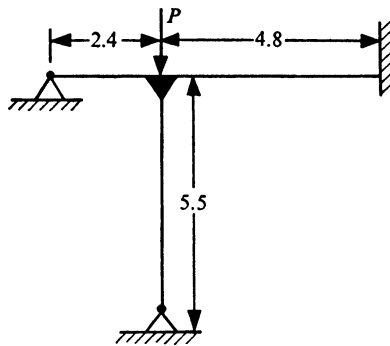


Fig. 4.27 Problem 4.10.

4.10 A 5.5-metre-long steel column is simply-supported at the bottom and rigidly connected to a horizontal beam at the top, as shown in Fig. 4.27. Determine the critical load for elastic buckling in the plane of the frame. The second moment of area for the column is $0.35 \times 10^{-4} \text{ m}^4$ and that for the beam is $0.275 \times 10^{-4} \text{ m}^4$. For steel, $E = 181.5 \text{ GPa}$.

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Elastic Buckling of Thin Plates

5.1 INTRODUCTION

In Chapters 1 to 4, we discussed the buckling of one-dimensional members of a structure. In doing so, we assumed that the cross-sectional dimensions of a member are such that the sectional behaviour is a function of only one variable. Also, we noted that, in a column, the buckling is normally confined to one vertical plane and, depending on the relative magnitudes of EI and GJ , the buckling mode is flexural (Chapter 1) or torsional (Chapter 3). Further, the number of boundary conditions is only four. Moreover, as stated in Chapter 3, the wide flanges of an open section column behave more like plate elements. The plates making up a column may undergo a form failure locally. Thus, when analyzing a column, the stability of plate elements has to be taken into account.

The buckling of a plate involves two planes, namely, XZ , YZ , and two boundary conditions on each edge of the plate. The basic difference between a column and a plate lies in the buckling characteristics. A column, once it buckles, cannot resist any additional axial load. Thus, the critical load of a column is also its failure load. On the other hand, a plate, since it is invariably supported at the edges (e.g., interconnection between two structural plates, and web connected to flanges), continues to resist the additional axial load even after the primary buckling load is reached and does not fail even when the load reaches a value 10–15 times the buckling load. Thus, for a plate, the postbuckling load (elastic) is much higher. When designing structural members, this fact is largely exploited to minimize the weight of the structure.

The local failure of the plate elements of a column represents only one aspect of plate instability. In a situation where the plate elements do not form part of a column (such as ship hull, aircraft wing, and web of plate girder), the plate instability has to be studied under different loading conditions (Bleich [1]). For example, the thin web of a plate girder may be subjected to a combination of axial and shear loading. Further, when a plate is a part of a structure, it is generally subjected to biaxial loading which can be constant or varying with respect to the X - or Y -axis.

In order to increase the buckling load of a plate, longitudinal and transverse stiffening is adopted (as, for example, in ship hull and aircraft wing). The analysis of such a stiffened structure is more involved because of the discontinuity in the plate rigidity along the X - and Y -axis.

Plates are of different shapes and sizes. Depending on the design requirements, the shape and size are selected. For example, generally, the end covers of a tank are circular, and the

wing skin of an aircraft is skew, in shape. Further, the ship hull is made of thick plates.

In this chapter, we shall give the buckling analysis of thin rectangular plates—unstiffened and stiffened—, thick rectangular plates, and circular plates.

5.2 EQUILIBRIUM APPROACH

To derive the equilibrium equation governing the buckling of a thin plate, we shall assume that the plate is subjected to various combinations of inplane forces. At buckling, a plate undergoes a small lateral deflection. This happens in the manner discussed when studying the behaviour of columns (Section 1.2.1).

Figure 5.1 shows the coordinate system for a plate and Fig. 5.2 all the possible internal stresses acting on a plate element. Assume that the thickness t of the plate is small as

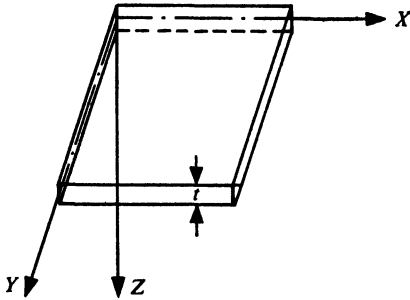


Fig. 5.1 Coordinate System for Plate.

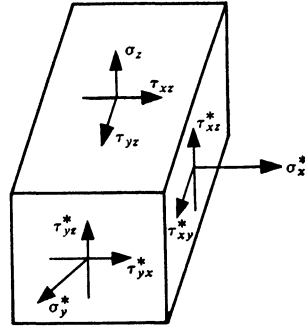


Fig. 5.2 Stresses Acting on Differential Element.

compared to its other two characteristic dimensions. We shall therefore treat this plate as subjected to plane stress, that is, the stresses are functions of only x and y . A quantity with an asterisk (Fig. 5.2) indicates the larger value of the normal or shear stress. For example,

$$\begin{aligned}\sigma_x^* &= \sigma_x + \frac{\partial}{\partial x} \sigma_x dx, \\ \tau_{xy}^* &= \tau_{xy} + \frac{\partial}{\partial x} \tau_{xy} dx, \\ \tau_{xz}^* &= \tau_{xz} + \frac{\partial}{\partial x} \tau_{xz} dx.\end{aligned}\tag{5.1}$$

We shall make the following two assumptions regarding the behaviour of a thin plate (Timoshenko and Krieger [2]):

(i) The effect of shear strains γ_{xz} and γ_{yz} on bending is negligible in a plate whose thickness is small as compared to the lateral dimensions. (As a consequence, the lines normal to the middle surface before bending remain straight and normal after bending.)

(ii) The normal stress σ_z and the corresponding strain ϵ_z are negligible. [Therefore, the

transverse deflection w at any point (x, y, z) is equal to the transverse deflection at the corresponding point $(x, y, 0)$ along the middle surface.]

When studying a plate, it is more convenient to work with stress resultants rather than with stresses. The stress resultants are defined as

$$\begin{aligned}
 N_x &= \int_{-t/2}^{+t/2} \sigma_x dz, & N_{yx} = N_{xy} &= \int_{-t/2}^{+t/2} \tau_{xy} dz, & N_y &= \int_{-t/2}^{+t/2} \sigma_y dz, \\
 M_x &= + \int_{-t/2}^{+t/2} \sigma_x z dz, & Q_x &= \int_{-t/2}^{+t/2} \tau_{xz} dz, \\
 M_y &= \int_{-t/2}^{+t/2} \sigma_y z dz, & Q_y &= \int_{-t/2}^{+t/2} \tau_{yz} dz, \\
 M_{yx} &= M_{xy} = \int_{-t/2}^{+t/2} \tau_{xy} z dz.
 \end{aligned} \tag{5.2}$$

The governing equation is obtained by using the static equations of equilibrium thus: (i) first, the inplane equilibrium is established, (ii) then, the equilibrium of bending moment and shear forces is considered, and (iii) finally, the results of (i) and (ii) are combined. The governing equation so obtained is in terms of the transverse displacement w . For a derivation of this equation, see also Gerard [3] and Chajes [4].

Inplane equilibrium

Consider now the differential inplane element shown in Fig. 5.2. The inplane forces acting on it are indicated in Fig. 5.3. Here, the quantities N_x^* , N_{xy}^* represent the values of the

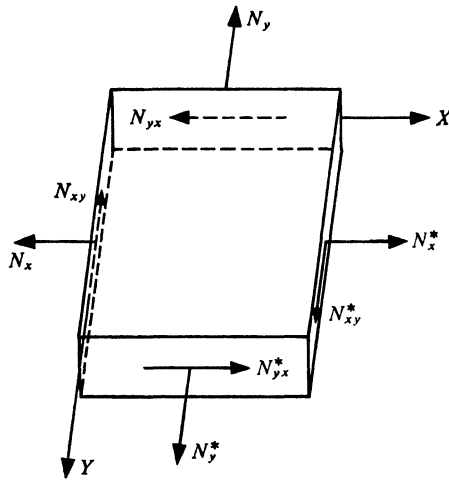


Fig. 5.3 Inplane Forces Acting on Element.

forces N_x , N_{xy} as we move along the X -direction, and N_y^* , N_{yx}^* represent the values of N_y ,

N_{yx} as we move along the Y -direction. These quantities can be obtained by replacing, in the first two equations in (5.1), σ_x by N_x , τ_{xy} by N_{xy} , and x by y . In view of the small deflection assumption, we neglect the change in the inplane forces due to bending. Summing up the forces along the X -direction, we get

$$N_x^* dy - N_x dy + N_{yx}^* dx - N_{yx} dx = 0. \quad (5.3)$$

Substituting for N_x^* and N_{yx}^* , we get

$$(N_x + \frac{\partial N_x}{\partial x} dx) dy - N_x dy + (N_{yx} + \frac{\partial N_{yx}}{\partial y} dy) dx - N_{yx} dx = 0. \quad (5.4)$$

Simplifying Eq. (5.4), we have

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0. \quad (5.5)$$

Similarly, summing up the forces in the Y -direction, we obtain

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0. \quad (5.6)$$

Transverse equilibrium

To obtain the equilibrium of forces in the Z -direction, we have to consider the deflected shape of the plate. Because of the transverse deflection, the forces N_x , N_y , N_{yx} , and N_{xy} will have components along the Z -direction. Figure 5.4 shows the deflected shape of the element along with the component forces $\partial w^*/\partial x$ and $\partial w^*/\partial y$ defined as

$$\begin{aligned} \frac{\partial w^*}{\partial x} &= \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) dx, \\ \frac{\partial w^*}{\partial y} &= \frac{\partial w}{\partial y} + \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} \right) dy. \end{aligned} \quad (5.7)$$

In view of the small deformation assumption,

$$\begin{aligned} \sin \left(\frac{\partial w}{\partial x} \right) &\approx \frac{\partial w}{\partial x}, \\ \cos \left(\frac{\partial w}{\partial x} \right) &\approx 1. \end{aligned} \quad (5.8)$$

The component of the forces N_x and N_x^* along the positive Z -axis is

$$(N_x + \frac{\partial N_x}{\partial x} dx) \left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right) dy - N_x \frac{\partial w}{\partial x} dy.$$

By retaining terms involving N_x , we can simplify these terms to

$$N_x \frac{\partial^2 w}{\partial x^2} dx dy.$$

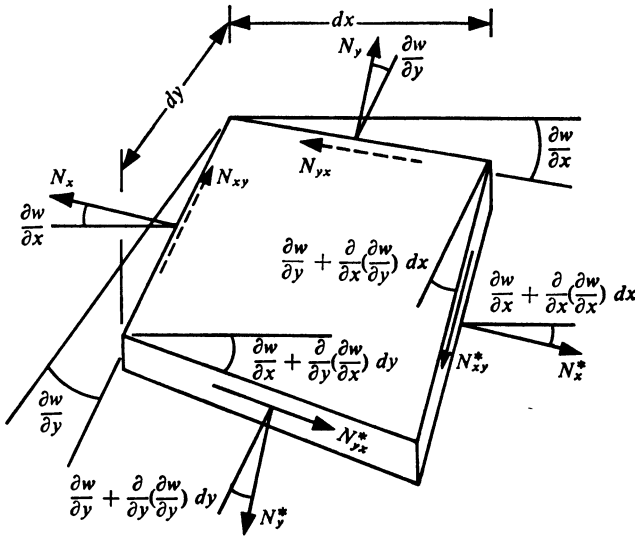


Fig. 5.4 Inplane Forces Acting on Deflected Plate Element.

The component of the forces N_y and N_y^* along the positive Z -axis is

$$(N_y + \frac{\partial}{\partial y} N_y dy) (\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy) dx - N_y^* \frac{\partial w}{\partial y} dx$$

which, on simplification, reduces to

$$N_y \frac{\partial^2 w}{\partial y^2} dx dy.$$

The z -component of the forces N_{xy} and N_{xy}^* is

$$(N_{xy} + \frac{\partial}{\partial x} N_{xy} dx) (\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} dx) dy - N_{xy}^* \frac{\partial w}{\partial y} dy.$$

This, on simplification, reduces to

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy.$$

Similarly, the component of the force N_{yx} along the Z -direction is

$$N_{yx} \frac{\partial^2 w}{\partial x \partial y} dx dy.$$

The equilibrium of moments about the Z -axis yields $N_{yx} = N_{xy}$. Hence, by adding all the

components of forces, we get the resultant force along the Z-axis due to the inplane forces as

$$(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}) dx dy. \quad (5.9)$$

To establish the contribution of the shear forces along the Z-direction, consider Fig. 5.5 which shows the moments and shear forces acting on the plate element. The shear forces

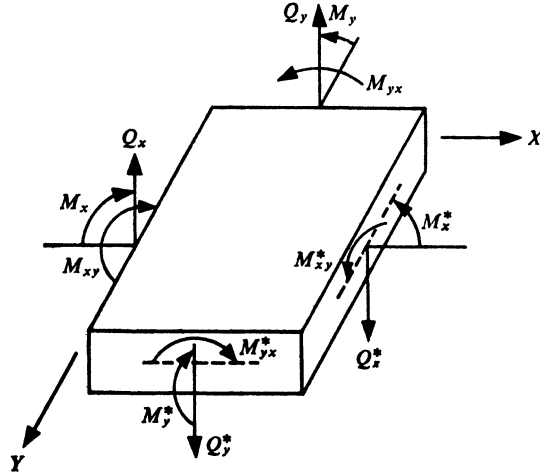


Fig. 5.5 Shear Forces and Moments Acting on Plate Element.

and moments on the x and y faces at a distance dx and dy from the origin are defined as

$$Q_x^* = Q_x + \frac{\partial}{\partial x} Q_x dx,$$

$$Q_y^* = Q_y + \frac{\partial}{\partial y} Q_y dy,$$

$$M_x^* = M_x + \frac{\partial}{\partial x} M_x dx,$$

$$M_y^* = M_y + \frac{\partial}{\partial y} M_y dy,$$

$$M_{xy}^* = M_{xy} + \frac{\partial}{\partial x} M_{xy} dx,$$

$$M_{yx}^* = M_{yx} + \frac{\partial}{\partial y} M_{yx} dy.$$

(5.10)

The component of the shear forces along the Z-direction is

$$Q_x^* dy - Q_x dy + Q_y^* dx - Q_y dx.$$

Substituting for Q_x^* and Q_y^* and simplifying, we obtain

$$\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right) dx dy. \quad (5.11)$$

Combining Eqs. (5.9) and (5.11), we get the equilibrium of forces along the Z -direction as

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial y^2} + N_y \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0. \quad (5.12)$$

Moment equilibrium

By taking the moments about the X -axis, we obtain

$$M_y^* dx - M_x dx - Q_y^* dx dy - Q_x^* dy \frac{dy}{2} + Q_x dy \frac{dy}{2} - M_{xy}^* dy + M_{xy} dy = 0. \quad (5.13)$$

Substituting for M_y^* , M_{xy}^* , and Q_y^* from Eqs. (5.10) in Eq. (5.13) and simplifying, we get

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0. \quad (5.14)$$

Taking the moments about the Y -axis, we have

$$M_x^* dy - M_x dy - Q_x^* dy dx - M_{yx}^* dx + M_{yx} dx - Q_y^* dx \frac{dx}{2} + Q_y^* dx \frac{dx}{2} = 0. \quad (5.15)$$

Substituting for M_x^* , M_{yx}^* , and Q_x^* from Eqs. (5.10) in Eq. (5.15) and simplifying, we obtain

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_x = 0. \quad (5.16)$$

Equations (5.12), (5.14), and (5.16) are the governing equations for a plate. These can be combined into a single equation by substituting for $\partial Q_x/\partial x$ and $\partial Q_y/\partial y$ from Eqs. (5.16) and (5.14) in Eq. (5.12). Differentiating Eqs. (5.14) and (5.16) with respect to dy and dx , respectively, we get

$$\frac{\partial^2 M_{xy}}{\partial y \partial x} - \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial Q_y}{\partial y} = 0, \quad (5.17)$$

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{yx}}{\partial y \partial x} - \frac{\partial Q_x}{\partial x} = 0. \quad (5.18)$$

Making use of the fact $M_{yx} = -M_{xy}$ and substituting Eqs. (5.17) and (5.18) in Eq. (5.12), we arrive at the governing equation for buckling, i.e.,

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0. \quad (5.19)$$

The moments M_x , M_y , and M_{xy} here can be replaced in terms of the curvature by making use of the constitutive equations and the strain-displacement relations. The constitutive

equations for an elastic plane stress problem are given by [see Eqs. (I.15) in Appendix I]

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y), \quad (5.20a)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x), \quad (5.20b)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}. \quad (5.20c)$$

In view of assumption (i), and the fact that the middle surface remains unstrained during the transverse displacement w , the displacements u and v along the X - and Y -direction at a distance z above the middle surface can be written as

$$u = -z \frac{\partial w}{\partial x}, \quad (5.21a)$$

$$v = -z \frac{\partial w}{\partial y}. \quad (5.21b)$$

The strain-displacement relations for a linear problem in the Cartesian coordinate system are (Timoshenko and Goodier [5])

$$\epsilon_x = \partial u / \partial x, \quad (5.22a)$$

$$\epsilon_y = \partial v / \partial y, \quad (5.22b)$$

$$\gamma_{xy} = \partial u / \partial y + \partial v / \partial x. \quad (5.22c)$$

Hence, the strains ϵ_x , ϵ_y , and γ_{xy} in terms of w can be written as

$$\epsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \quad (5.23a)$$

$$\epsilon_y = -z \frac{\partial^2 w}{\partial y^2}, \quad (5.23b)$$

$$\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}. \quad (5.23c)$$

Substituting for strains from Eqs. (5.23) in Eqs. (5.20), we can write the stress-curvature relations as

$$\sigma_x = -\frac{Ez}{1-\nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \quad (5.24a)$$

$$\sigma_y = -\frac{Ez}{1-\nu^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \quad (5.24b)$$

$$\tau_{xy} = -\frac{Ez}{1+\nu} \frac{\partial^2 w}{\partial x \partial y}. \quad (5.24c)$$

Substituting for σ_x , σ_y , and τ_{xy} in the corresponding equations in (5.2) and carrying out the integration, the moment-curvature relations we obtain are

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \quad (5.25a)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), \quad (5.25b)$$

$$M_{xy} = -M_{yx} = -D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y}, \quad (5.25c)$$

where

$$D = Et^3/[12(1 - \nu^2)].$$

Here, D is the flexural rigidity per unit length of the plate. This is analogous to the bending stiffness EI of a beam (see Chapter 1). If we compare the rigidity of a plate with the rigidity of a beam having the same width and depth as the plate, we find that a plate strip is stiffer than the beam by a factor $1/(1 - \nu^2)$. This is so because a beam is free to deform laterally unlike a plate strip which is restrained due to the presence of adjacent strips.

Substituting Eqs. (5.25) in Eq. (5.19) and assuming that the plate is of constant thickness, the governing equation we obtain is

$$D\nabla^4 w = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}, \quad (5.26)$$

where

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}.$$

It should be noted that N_x in the foregoing discussion is positive for tensile loading. In buckling, N_x and N_y are generally compressive. Incorporating the proper sign in Eq. (5.26), we get

$$D\nabla^4 w + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0. \quad (5.27)$$

For a nonuniform plate (i.e., plate thickness is a function of x and/or y but is symmetric with respect to the Z -axis), the governing equation, as we shall now discover, is different and the process of deriving it is quite involved. Here, the change occurs when Eqs. (5.25) are substituted in Eq. (5.19). Assuming D is a function of x and y , we have

$$\begin{aligned} \frac{\partial M_x}{\partial x} &= \frac{\partial}{\partial x} \left[-D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] \\ &= -\frac{\partial D}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) - D \left(\frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial x \partial y^2} \right). \end{aligned}$$

Hence,

$$\frac{\partial^2 M_x}{\partial x^2} = -\frac{\partial^2 D}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) - 2 \frac{\partial D}{\partial x} \left(\frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial x \partial y^2} \right) - D \left(\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right). \quad (5.28)$$

Similarly, it can be shown that

$$\frac{\partial^2 M_y}{\partial y^2} = -\frac{\partial^2 D}{\partial y^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) - 2 \frac{\partial D}{\partial y} \left(\frac{\partial^3 w}{\partial y^3} + \nu \frac{\partial^3 w}{\partial y \partial x^2} \right) - D \left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right), \quad (5.29)$$

$$\begin{aligned} \frac{\partial^2 M_{xy}}{\partial x \partial y} = & -(1 - \nu) \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} - (1 - \nu) \frac{\partial D}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} \\ & - (1 - \nu) \frac{\partial D}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} - D(1 - \nu) \frac{\partial^4 w}{\partial x^2 \partial y^2}. \end{aligned} \quad (5.30)$$

Substituting Eqs. (5.28), (5.29), and (5.30) in Eq. (5.19), we get

$$\begin{aligned} D \nabla^4 w + \frac{\partial^2 D}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2 D}{\partial y^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) + 2 \frac{\partial D}{\partial x} \left(\frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial x \partial y^2} \right) \\ + 2 \frac{\partial D}{\partial y} \left(\frac{\partial^3 w}{\partial y^3} + \nu \frac{\partial^3 w}{\partial x^2 \partial y} \right) + (1 - \nu) \frac{\partial D}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} + (1 - \nu) \frac{\partial D}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} \\ + 2(1 - \nu) \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0. \end{aligned} \quad (5.31)$$

Equation (5.31) is applicable for a rectangular nonuniform plate, symmetric with respect to the Z-axis.

Boundary conditions

Equation (5.27) or (5.31) is a fourth order partial differential equation in x and y , and hence, for a unique solution, eight boundary conditions—four along x and four along y —are required. The possible ideal boundary conditions are as follows:

(i) Simply-supported or hinged ends. Here,

$$w = 0 \quad (x = 0, a), \quad (5.32a)$$

$$w = 0 \quad (y = 0, b). \quad (5.32b)$$

As the edges are free to rotate, the moment M_x or M_y , must be zero. Hence, from Eqs. (5.25a) and (5.25b),

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = 0 \quad (x = 0, a), \quad (5.33a)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = 0 \quad (y = 0, b). \quad (5.33b)$$

(ii) Clamped edges. Here,

$$w = 0 \quad (x = 0, a), \quad (5.34a)$$

$$w = 0 \quad (y = 0, b). \quad (5.34b)$$

As the edges cannot rotate, the first derivative of w with respect to x and y must be zero. That is,

$$\partial w / \partial x = 0 \quad (x = 0, a), \quad (5.35a)$$

$$\partial w / \partial y = 0 \quad (y = 0, b). \quad (5.35b)$$

(iii) Free edges. The boundary conditions here are quite involved. In Fig. 5.5, we see that, on any edge, there exist two moments and the shear force. All of these must be zero for a free edge. This results in a total of 12 boundary conditions, thereby implying that the problem is overdetermined since we need only eight boundary conditions for a unique solution. Timoshenko and Krieger [2] have shown that these three quantities can be reduced to two by combining the shearing force and the twisting moment. The boundary conditions (Kirchhoff's boundary conditions) to be satisfied for a free edge are then

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = 0 \quad (x = 0, a), \quad (5.36a)$$

$$\frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} = 0 \quad (x = 0, a). \quad (5.36b)$$

Similarly,

$$\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = 0 \quad (y = 0, b), \quad (5.37a)$$

$$\frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial y \partial x^2} = 0 \quad (y = 0, b). \quad (5.37b)$$

We shall now demonstrate the use of Eq. (5.27) in estimating the critical load of uniform rectangular plates.

EXAMPLE 5.1 (Buckling load of rectangular plate axially compressed in one direction) Consider a simply-supported rectangular plate (Fig. 5.6) subjected to a uniform axially compressive force N_x per unit length along the edges $x = 0, a$. Putting N_y and N_{xy} equal to zero in Eq. (5.27), we get the governing equation as

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2} = 0. \quad (5.38)$$

In terms of the nondimensional parameters ξ and η , defined as

$$x = a\xi, \quad y = b\eta, \quad (5.39)$$

Eq. (5.38) can be written as

$$\frac{\partial^4 w}{\partial \xi^4} + \frac{2a^2}{b^2} \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{a^4}{b^4} \frac{\partial^4 w}{\partial \eta^4} + \frac{N_x a^2}{D} \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.40)$$

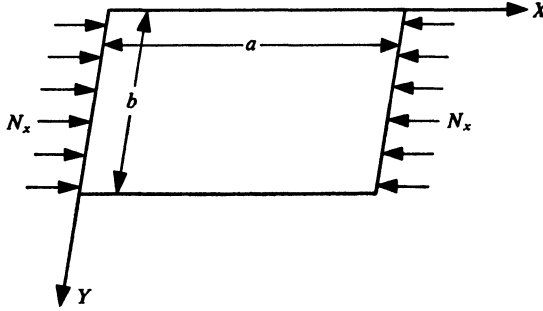


Fig. 5.6 Rectangular Plate under Axial Load.

Defining $p (=a/b)$ as the aspect ratio of plate and

$$\lambda^2 = N_x \frac{a^2}{D} = \sigma_x \frac{t a^2}{E t^3} \{12(1 - \nu^2)\}, \quad (5.41)$$

we can write Eq. (5.40) as

$$\frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \lambda^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.42)$$

Since all the edges are simply-supported, the boundary conditions are given by (5.32) and (5.33). In a nondimensional form, these conditions can be written as

$$w = \frac{\partial^2 w}{\partial \xi^2} + \nu p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\xi = 0, 1), \quad (5.43)$$

$$w = p^2 \frac{\partial^2 w}{\partial \eta^2} + \nu \frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\eta = 0, 1). \quad (5.44)$$

Since $w = 0$ along $\xi = 0, 1$ and $\eta = 0, 1$, we have

$$\partial^2 w / \partial \eta^2 = 0 \quad (\xi = 0, 1),$$

$$\partial^2 w / \partial \xi^2 = 0 \quad (\eta = 0, 1).$$

Making use of these relations, we find the boundary conditions (5.43) and (5.44) reduce to

$$w = \partial^2 w / \partial \xi^2 = 0 \quad (\xi = 0, 1), \quad (5.45)$$

$$w = \partial^2 w / \partial \eta^2 = 0 \quad (\eta = 0, 1). \quad (5.46)$$

The differential equation (5.42) and the boundary conditions (5.45) and (5.46) contain terms which are of even order in ξ and η ; hence, we seek a variable separable type of solution. Thus, for Eq. (5.42), assume a solution of the form

$$w(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin m\pi\xi \sin n\pi\eta \quad (m = 1, 2, \dots, n = 1, 2, \dots), \quad (5.47)$$

where m and n define the number of half waves that the plate buckles in the X - and Y -direction, respectively, and A_{mn} represents the amplitudes of the shape function or the mode shapes. Equation (5.47) satisfies the boundary conditions exactly. Substituting Eq. (5.47) in Eq. (5.42), we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}(m^4\pi^4 + 2p^2m^2n^2\pi^4 + p^4n^4\pi^4 - \lambda^2m^2\pi^2) \sin m\pi\xi \sin n\pi\eta = 0. \quad (5.48)$$

The terms within the parentheses in Eq. (5.48) consist of a sum of an infinite number of positive functions. The only way such a sum can be zero is when the coefficient of each one of the terms is equal to zero. Thus,

$$A_{mn}(m^4\pi^4 + 2p^2m^2n^2\pi^4 + p^4n^4\pi^4 - \lambda^2m^2\pi^2) = 0. \quad (5.49)$$

Expression (5.49) can be satisfied if either $A_{mn} = 0$ or the quantity within the parentheses is zero. The first alternative leads to a trivial solution, where, for all loads, the plate remains flat. The nontrivial solution gives

$$\lambda^2 = \frac{N_x}{D}a^2 = m^2\pi^2 + 2p^2n^2\pi^2 + p^4\frac{n^4}{m^2}\pi^2, \quad (5.50)$$

$$\begin{aligned} N_x &= \frac{D\pi^2}{a^2}(m^2 + 2p^2n^2 + p^4\frac{n^4}{m^2}) \\ &= \frac{D\pi^2}{b^2}(\frac{m}{p} + \frac{n^2}{m}p)^2. \end{aligned} \quad (5.51)$$

Since we require to know the lowest value of N_x at which the equilibrium of the plate changes from the flat to a bent configuration, we need to determine the values of m and n that minimize Eq. (5.51). It is easy to visualize that as n increases, the critical load N_x also increases; hence, for the lowest value of N_x , n must be equal to one. This implies that the plate buckles with one half sine wave along the Y -direction. The number of half waves along the X -direction that correspond to the minimum value of N_x can be found by taking the derivative of Eq. (5.51) with respect to m , with n equal to 1, and equating the resulting expression to zero. Thus,

$$\frac{dN_x}{dm} = \frac{2D\pi^2}{b^2}(\frac{m}{p} + \frac{p}{m})(\frac{1}{p} - \frac{p}{m^2}) = 0. \quad (5.52)$$

This yields

$$m = p. \quad (5.53)$$

Substituting this value of m in Eq. (5.51), keeping n equal to 1, we get

$$(N_x)_{cr} = 4D\pi^2/b^2. \quad (5.54)$$

Thus, a simply-supported plate buckles with one half wave in the Y -direction and p half waves in the X -direction, i.e., p must be an integer. This implies that the plate buckles into square plates. For noninteger values of p , the buckling load is higher than that for the

integer values. For such a case, Eq. (5.51) can be written as

$$(N_x)_{cr} = kD\pi^2/b^2, \quad (5.55)$$

where

$$k = (m/p + n^2p/m)^2. \quad (5.56)$$

Figure 5.7 shows the variation of k with the aspect ratio p . For $p < 1$, k varies considerably with the aspect ratio. However, for $p \geq 4$, the variation of k is negligible, and hence we can fairly accurately take the value of $k = 4.0$ for $p > 4.0$. The values $m = 1, 2, \dots$ indicate the number of half waves that the plate will buckle in the X -direction for a given aspect ratio. It can be observed from Fig. 5.7 that, for some values of the aspect ratio, there

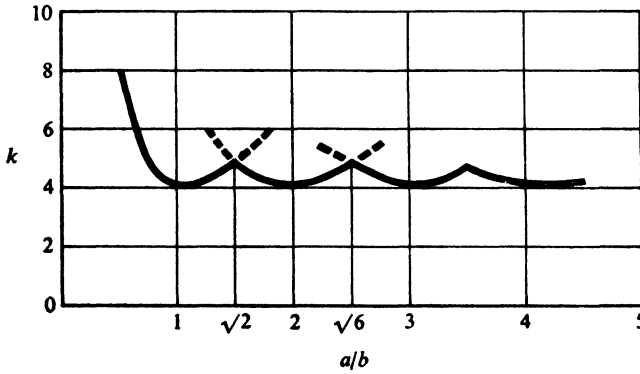


Fig. 5.7 Variation of Buckling Load Coefficient k with Aspect Ratio.

is a transition from m to $m + 1$ half waves. At these points, the buckling load coefficient k is the same, i.e.,

$$m/p + p/m = (m + 1)/p + p/(m + 1).$$

From this equation, we obtain

$$p = \sqrt{m(m + 1)}. \quad (5.57)$$

Setting $m = 1$, we get $p = \sqrt{2}$. At this ratio, we have the transition from one to two half waves. Similarly, for $p = \sqrt{6}, \sqrt{12}, \dots$, we have the transition from two to three half waves, three to four half waves, $\dots$. Further, for $p < \sqrt{2}$, the plate will buckle with one half wave in the X -direction, for $\sqrt{2} < p < \sqrt{6}$, the plate will buckle with two half waves, and so on.

The critical stress σ_x per unit length is given as

$$(\sigma_x)_{cr} = \frac{(N_x)_{cr}}{t} = \frac{kE}{12(1 - \nu^2)} \left(\frac{t}{b}\right)^2. \quad (5.58)$$

Equation (5.58) can be rewritten as

$$(\sigma_x)_{cr} = KE(t/b)^2, \quad (5.59)$$

where $K = k/[12(1 - \nu^2)]$ and, for $\nu = 0.3$, $K = 3.62$ and $k = 4.0$. If we compare Eq. (5.59) with Eq. (1.12), we find that the critical stress of a column is inversely proportional to the length of the column, whereas, for a plate, the critical stress is inversely proportional to the width of the plate (Timoshenko and Gere [6]).

Table 5.1 shows the variation of critical stress with the aspect ratio for a plate of aluminium alloy.

Table 5.1 Values of $(\sigma_x)_{cr}$ for Simply-Supported Rectangular Axially Compressed Plate

Given

$$E = 0.588 \times 10^{11} \text{ N/m}^2, \quad t/b = 0.02, \quad \nu = 0.3.$$

p	0.2	0.4	0.8	1.0	$\sqrt{2}$
k	27.04	8.41	4.20	4.0	4.49
σ_{cr}	0.5751×10^9	0.1788×10^9	0.0893×10^9	0.0850×10^9	0.0955×10^9
p	2.0	$\sqrt{6}$	3.2	4.5	$\sqrt{12}$
k	4.0	4.17	4.016	4.055	4.08
σ_{cr}	0.085×10^9	0.0886×10^9	0.0854×10^9	0.0862×10^9	0.0868×10^9

EXAMPLE 5.2 (Buckling of rectangular plate with loading edges simply-supported and other two edges clamped) The situation we consider here is a little more complicated than that in Example 5.1. Let the edges $\xi = 0, 1$ be simply-supported, and the edges $\eta = 0, 1$ be clamped. The governing equation here is (5.42), i.e.,

$$\frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \lambda^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.42)$$

Further, the boundary conditions in the nondimensional form are

$$w = \frac{\partial^2 w}{\partial \xi^2} + \nu p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\xi = 0, 1), \quad (5.60)$$

$$w = \partial w / \partial \eta = 0 \quad (\eta = 0, 1). \quad (5.61)$$

Following the same steps as in Example 5.1, we here choose the solution for Eq. (5.42) as

$$w(\xi, \eta) = F(\eta) \sin m\pi\xi. \quad (5.62)$$

The amplitude term is included in $F(\eta)$ which we have yet to determine. Expression (5.62) satisfies the boundary conditions along $\xi = 0, 1$. Now, substituting relation (5.62) in Eq. (5.42), the ordinary differential equation for determining $F(\eta)$ we get is

$$\left(\frac{d^4 F}{d\eta^4} - 2\frac{m^2\pi^2}{p^2}\frac{d^2 F}{d\eta^2} + \frac{m^4\pi^4}{p^4}F - \lambda^2\frac{m^2\pi^2}{p^4}F\right) \sin m\pi\xi = 0. \quad (5.63)$$

Since $\sin m\pi\xi$ is not zero, the quantity within the parentheses must be zero. The general solution for Eq. (5.63) can be written as

$$F(\eta) = A_1 \cosh \alpha\eta + A_2 \sinh \alpha\eta + A_3 \cos \beta\eta + A_4 \sin \beta\eta, \quad (5.64)$$

where

$$\begin{aligned} \alpha &= \left[\frac{m^2\pi^2}{p^2} + \left(\lambda^2\frac{m^2\pi^2}{p^4}\right)^{1/2}\right]^{1/2}, \\ \beta &= \left[-\frac{m^2\pi^2}{p^2} + \left(\lambda^2\frac{m^2\pi^2}{p^4}\right)^{1/2}\right]^{1/2}. \end{aligned} \quad (5.65)$$

The constants A_1, A_2, A_3 , and A_4 have to be determined from the edge conditions (5.61). After transforming these conditions in terms of $F(\eta)$ from Eq. (5.62), the set of four homogeneous algebraic equations we obtain is

$$A_1 + A_3 = 0, \quad (5.66a)$$

$$A_2\alpha + A_4\beta = 0, \quad (5.66b)$$

$$A_1 \cosh \alpha + A_2 \sinh \alpha + A_3 \cos \beta + A_4 \sin \beta = 0, \quad (5.66c)$$

$$A_1\alpha \sinh \alpha + A_2\alpha \cosh \alpha - A_3\beta \sin \beta + A_4\beta \cos \beta = 0. \quad (5.66d)$$

Solving these, we get

$$A_1(\cosh \alpha - \cos \beta) + A_2(\sinh \alpha - \frac{\alpha}{\beta} \sin \beta) = 0, \quad (5.67a)$$

$$A_1(\alpha \sinh \alpha + \beta \sin \beta) + A_2(\alpha \cosh \alpha - \alpha \cos \beta) = 0. \quad (5.67b)$$

For a nontrivial solution, the determinant of the coefficients of A_1 and A_2 must vanish. That is,

$$\begin{vmatrix} \cosh \alpha - \cos \beta & \sinh \alpha - \frac{\alpha}{\beta} \sin \beta \\ \alpha \sinh \alpha + \beta \sin \beta & \alpha \cosh \alpha - \alpha \cos \beta \end{vmatrix} = 0. \quad (5.68)$$

This leads to the characteristic equation

$$2(1 - \cosh \alpha \cos \beta) = \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) \sinh \alpha \sin \beta. \quad (5.69)$$

For a given plate material and aspect ratio, we can now obtain the critical stress from

Eq. (5.69). The critical stress can also be written as

$$(\sigma_x)_{cr} = \frac{kE}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2. \quad (5.58)$$

The value of k here is different from that of k in Example 5.1.

Table 5.2 shows the variation of critical stress with the aspect ratio for a plate of aluminium alloy.

Table 5.2 Values of $(\sigma_x)_{cr}$ for Axially Compressed Plate, Two Opposite Edges of Which Are Simply-Supported and Other Two Clamped

Given

$$E = 0.588 \times 10^{11} \text{ N/m}^2, \quad t/b = 0.02, \quad \nu = 0.3.$$

p	0.2	0.4	0.8	1.0	$\sqrt{2}$
k	27.86	9.49	7.44	7.69	7.04
σ_{cr}	0.5927×10^9	0.2021×10^9	0.1583×10^9	0.1638×10^9	0.1498×10^9
p	2.0	$\sqrt{6}$	3.2	$\sqrt{12}$	4.5
k	6.99	7.02	6.98	6.99	6.98
σ_{cr}	0.1487×10^9	0.1494×10^9	0.1486×10^9	0.1487×10^9	0.1485×10^9

EXAMPLE 5.3 (Buckling of rectangular plate with loading edges simply-supported and one edge clamped and other free) Consider the rectangular plate shown in Fig. 5.6. Assume that the edge $\eta = 0$ is built-in, and the edge $\eta = 1$ is free. The differential equation governing the plate, as before, is

$$\frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \lambda^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.42)$$

The boundary conditions in the nondimensional form are obtained after transforming Eqs. (5.34b), (5.35b), (5.37a), and (5.37b). Thus,

$$w = 0 \quad (\eta = 0), \quad (5.70a)$$

$$\partial w / \partial \eta = 0 \quad (\eta = 0), \quad (5.70b)$$

$$p^2 \frac{\partial^2 w}{\partial \eta^2} + \nu \frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\eta = 1), \quad (5.70c)$$

$$p^2 \frac{\partial^3 w}{\partial \eta^3} + (2 - \nu) \frac{\partial^3 w}{\partial \eta \partial \xi^2} = 0 \quad (\eta = 1). \quad (5.70d)$$

Since the two opposite edges are simply-supported, the general solution we chose for $F(\eta)$ in Example 5.2 is still valid. Further, the constants A_1, A_2, A_3 , and A_4 in Eq. (5.64) are obtained by using the boundary conditions (5.70) after transforming the latter in terms of $F(\eta)$. The boundary conditions in terms of $F(\eta)$ are

$$F(\eta) = 0 \quad (\eta = 0), \quad (5.71a)$$

$$dF/d\eta = 0 \quad (\eta = 0), \quad (5.71b)$$

$$p^2 \frac{d^2 F}{d\eta^2} - \nu m^2 \pi^2 F = 0 \quad (\eta = 1), \quad (5.71c)$$

$$p^2 \frac{d^3 F}{d\eta^3} - (2 - \nu) m^2 \pi^2 \frac{dF}{d\eta} = 0 \quad (\eta = 1). \quad (5.71d)$$

Using conditions (5.71) along with Eq. (5.64), we get the set of four simultaneous homogeneous algebraic equations

$$A_1 + A_3 = 0, \quad (5.72a)$$

$$A_2 \alpha + A_4 \beta = 0, \quad (5.72b)$$

$$A_1 \alpha^2 \cosh \alpha + A_2 \alpha^2 \sinh \alpha - A_3 \beta^2 \cos \beta - A_4 \beta^2 \sin \beta - \nu \frac{m^2 \pi^2}{p^2} (A_1 \cosh \alpha + A_2 \sinh \alpha + A_3 \cos \beta + A_4 \sin \beta) = 0, \quad (5.72c)$$

$$A_1 \alpha^3 \sinh \alpha + A_2 \alpha^3 \cosh \alpha + A_3 \beta^3 \sin \beta - A_4 \beta^3 \cos \beta - (2 - \nu) \frac{m^2 \pi^2}{p^2} (A_1 \alpha \sinh \alpha + A_2 \alpha \cosh \alpha - A_3 \beta \sin \beta + A_4 \beta \cos \beta) = 0. \quad (5.72d)$$

Equations (5.72) can be simplified to

$$A_1 [\cosh \alpha (\alpha^2 - \nu \frac{m^2 \pi^2}{p^2}) + \cos \beta (\beta^2 + \nu \frac{m^2 \pi^2}{p^2})] + A_2 [\sinh \alpha (\alpha^2 - \nu \frac{m^2 \pi^2}{p^2}) + \frac{\alpha}{\beta} \sin \beta (\beta^2 + \nu \frac{m^2 \pi^2}{p^2})] = 0, \quad (5.73a)$$

$$A_1 [\alpha \sinh \alpha \{\alpha^2 - (2 - \nu) \frac{m^2 \pi^2}{p^2}\} - \beta \sin \beta \{\beta^2 + (2 - \nu) \frac{m^2 \pi^2}{p^2}\}] + A_2 [\alpha \cosh \alpha \{\alpha^2 - (2 - \nu) \frac{m^2 \pi^2}{p^2}\} + \alpha \cos \beta \{\beta^2 + (2 - \nu) \frac{m^2 \pi^2}{p^2}\}] = 0. \quad (5.73b)$$

Defining

$$\bar{s} = \alpha^2 - \nu \frac{m^2 \pi^2}{p^2} = \beta^2 + (2 - \nu) \frac{m^2 \pi^2}{p^2}, \quad (5.74a)$$

$$\bar{t} = \beta^2 + \nu \frac{m^2 \pi^2}{p^2} = \alpha^2 - (2 - \nu) \frac{m^2 \pi^2}{p^2}, \quad (5.74b)$$

we can write Eqs. (5.73) as

$$A_1(\bar{s} \cosh \alpha + \bar{t} \cos \beta) + A_2(\bar{s} \sinh \alpha + \frac{\alpha \bar{t}}{\beta} \sin \beta) = 0, \quad (5.75a)$$

$$A_1(\alpha \bar{t} \sinh \alpha - \beta \bar{s} \sin \beta) + A_2(\bar{t} \alpha \cosh \alpha + \alpha \bar{s} \cos \beta) = 0. \quad (5.75b)$$

For a nontrivial solution, the determinant of the coefficients of A_1 and A_2 must vanish. That is,

$$\begin{vmatrix} \bar{s} \cosh \alpha + \bar{t} \cos \beta & \bar{s} \sinh \alpha + \frac{\alpha \bar{t}}{\beta} \sin \beta \\ \alpha \bar{t} \sinh \alpha - \beta \bar{s} \sin \beta & \alpha \bar{t} \cosh \alpha + \alpha \bar{s} \cos \beta \end{vmatrix} = 0. \quad (5.76)$$

This leads to the characteristic equation

$$2\bar{s}\bar{t} + (\bar{s}^2 + \bar{t}^2) \cosh \alpha \cos \beta = \left(\frac{\alpha^2 \bar{t}^2 - \beta^2 \bar{s}^2}{\alpha \beta} \right) \sinh \alpha \sin \beta. \quad (5.77)$$

For a given plate material, we can now obtain the critical load after solving for α and β which contain N_x . The critical stress can then be expressed by Eq. (5.58), i.e.,

$$(\sigma_x)_{cr} = \frac{kE}{12(1-\nu^2)} \left(\frac{t}{b} \right)^2. \quad (5.58)$$

Here again, the value of k will be different from the values of k in Examples 5.1 and 5.2.

Table 5.3 shows the variation of $(\sigma_x)_{cr}$ with the aspect ratio for a plate of aluminium alloy.

Table 5.3 Values of $(\sigma_x)_{cr}$ for Axially Compressed Plate, Two Opposite Edges of Which Are Simply-Supported and One Edge Clamped and Other Free

Given

$$E = 0.588 \times 10^{11} \text{ N/m}^2, \quad t/b = 0.02, \quad \nu = 0.3.$$

p	0.2	0.4	0.8	1.0	$\sqrt{2}$
k	25.35	6.34	2.16	1.75	1.43
σ_{cr}	0.5393×10^9	0.1348×10^9	0.0459×10^9	0.0341×10^9	0.0304×10^9
p	2.0	$\sqrt{6}$	3.2	$\sqrt{12}$	4.5
k	1.47	1.64	1.39	1.41	1.40
σ_{cr}	0.0318×10^9	0.0350×10^9	0.0297×10^9	0.0299×10^9	0.02973×10^9

The edge conditions considered in Examples 5.1 to 5.3 are referred to as the ideal edge

conditions. These conditions can exist if a plate is treated as an individual element. However, in practice, such situations are rare. Instead, a plate is usually connected to other members. For example, in an *I*-section, if we treat one side of the flange as a plate element, we find that it is connected elastically to the web; conversely, the web is connected elastically to the flange. In such a section, the buckling of the flange is restrained by the web and vice versa. Hence, for calculating the buckling load of a member of an *I*-section, the interaction of deformation between the adjoining members has to be taken into account.

We shall now consider examples to demonstrate the effect of elastic support.

EXAMPLE 5.4 (Buckling of plate elastically restrained along one edge and free at other edge) Consider the rectangular plate shown in Fig. 5.8. It is simply-supported along the

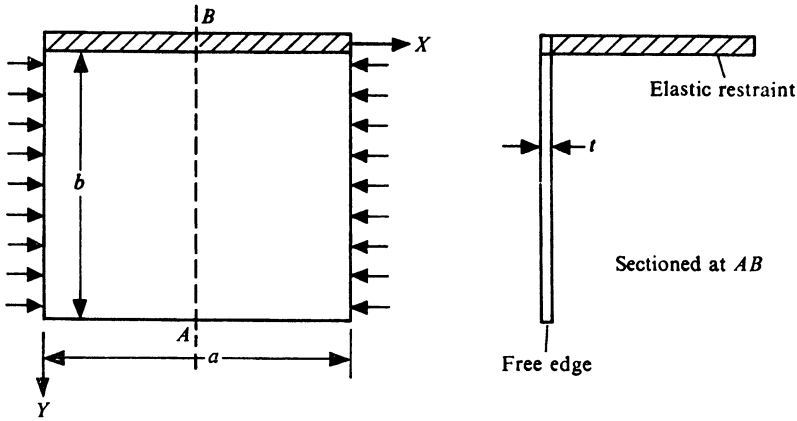


Fig. 5.8 Buckling of Plate with Elastic Restraint.

loaded edge, the edge $\eta = 0$ is elastically restrained, and the edge $\eta = 1$ is free. The general solution to the differential equation (5.42) is

$$w(\xi, \eta) = \sin m\pi\xi(A_1 \cosh \alpha\eta + A_2 \sinh \alpha\eta + A_3 \cos \beta\eta + A_4 \sin \beta\eta), \quad (5.78)$$

where α and β are as defined by Eqs. (5.65). The constants A_1 , A_2 , A_3 , and A_4 are determined from the boundary conditions which are

$$w = 0 \quad (\eta = 0), \quad (5.79a)$$

$$\phi = \bar{\phi} \quad (\eta = 0), \quad (5.79b)$$

$$p^2 \frac{\partial^2 w}{\partial \eta^2} + \nu \frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\eta = 1), \quad (5.79c)$$

$$p^2 \frac{\partial^3 w}{\partial \eta^3} + (2 - \nu) \frac{\partial^3 w}{\partial \eta \partial \xi^2} = 0 \quad (\eta = 1). \quad (5.79d)$$

Equation (5.79b) represents the condition of continuity and indicates that the angle of

rotation ϕ at the edge of the buckling plate is equal to the angle of rotation $\bar{\phi}$ at the edge of the restraining plate. This condition is necessary to ensure that the structure does not fail. To incorporate this condition in the analysis, ϕ and $\bar{\phi}$ have to be expressed in terms of the transverse deflection w . As we have already noted, the bending moment generated by a plate is M_y per unit length (see Fig. 5.5). For continuity, this moment also acts on the restraining plate which causes the rotation $\bar{\phi}$ of the plate. In accordance with the small deformation assumption,

$$M_y = -\bar{\delta}\bar{\phi}, \quad (5.80)$$

where $\bar{\delta}$ is a constant of proportionality and depends on the dimensions of the restraining plate. (Bleich [1] has given the expression for $\bar{\delta}$ for various types of restraining elements.) The moment M_y , in terms of w , generated by a plate is given by Eq. (5.25b). Since $w = 0$ along the edge $y = 0$ for all x , $\partial^2 w / \partial x^2 = 0$ everywhere along this edge. Hence,

$$M_y = -D \frac{\partial^2 w}{\partial y^2} \quad (y = 0). \quad (5.81)$$

Substituting for M_y in Eq. (5.80), we get

$$\bar{\phi} = \frac{D}{\bar{\delta}} \frac{\partial^2 w}{\partial y^2} \quad (y = 0). \quad (5.82)$$

Since the bending slope $\phi = \partial w / \partial y$ at $y = 0$, the boundary condition (5.79b) becomes

$$\frac{\partial w}{\partial y} - \frac{D}{\bar{\delta}} \frac{\partial^2 w}{\partial y^2} = 0 \quad (y = 0). \quad (5.83)$$

In terms of the nondimensional parameter η , this can be written as

$$\frac{\partial w}{\partial \eta} - \frac{D}{\bar{\delta}b} \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0)$$

or

$$\frac{\partial w}{\partial \eta} - \delta \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0), \quad (5.84)$$

where

$$\delta = D/(\bar{\delta}b) \quad (5.85)$$

is defined as the coefficient of restraint. δ , in general, can vary from 0 to ∞ . When $\delta = 0$, the plate is completely clamped, and when $\delta = \infty$, the plate is simply-supported along the edges. Substituting Eq. (5.78) along with its derivatives in Eqs. (5.79a), (5.84), (5.79c), and (5.79d), we get the set of equations

$$A_1 + A_3 = 0, \quad (5.86a)$$

$$A_2\alpha + A_4\beta = \delta(A_1\alpha^2 - A_3\beta^2), \quad (5.86b)$$

$$A_1 \alpha^2 \cosh \alpha + A_2 \alpha^2 \sinh \alpha - A_3 \beta^2 \cos \beta - A_4 \beta^2 \sin \beta - \nu \frac{m^2 \pi^2}{p^2} (A_1 \cosh \alpha + A_2 \sinh \alpha + A_3 \cos \beta + A_4 \sin \beta) = 0, \quad (5.86c)$$

$$A_1 \alpha^3 \sinh \alpha + A_2 \alpha^3 \cosh \alpha + A_3 \beta^3 \sin \beta - A_4 \beta^3 \cos \beta - (2 - \nu) \frac{m^2 \pi^2}{p^2} (A_1 \alpha \sinh \alpha + A_2 \alpha \cosh \alpha - A_3 \beta \sin \beta + A_4 \beta \cos \beta) = 0. \quad (5.86d)$$

Making use of Eqs. (5.86a) and (5.86b), we eliminate A_3 and A_4 from Eqs. (5.86c) and (5.86d). The resulting equations are

$$A_1 [\cosh \alpha (\alpha^2 - \nu \frac{m^2 \pi^2}{p^2}) + \cos \beta (\beta^2 + \nu \frac{m^2 \pi^2}{p^2}) - \sin \beta \frac{\delta}{\beta} (\alpha^2 + \beta^2) (\beta^2 + \nu \frac{m^2 \pi^2}{p^2})] + A_2 [\sinh \alpha (\alpha^2 - \nu \frac{m^2 \pi^2}{p^2}) + \frac{\alpha}{\beta} \sin \beta (\beta^2 + \nu \frac{m^2 \pi^2}{p^2})] = 0, \quad (5.87a)$$

$$A_1 [\alpha \sinh \alpha \{ \alpha^2 - (2 - \nu) \frac{m^2 \pi^2}{p^2} \} - \beta \sin \beta \{ \beta^2 + (2 - \nu) \frac{m^2 \pi^2}{p^2} \} - \beta \cos \beta \{ \beta^2 + (2 - \nu) \frac{m^2 \pi^2}{p^2} \} \frac{\delta}{\beta} (\alpha^2 + \beta^2)] + A_2 [\alpha \cosh \alpha \{ \alpha^2 - (2 - \nu) \frac{m^2 \pi^2}{p^2} \} + \frac{\alpha}{\beta} \cos \beta \{ \beta^2 + (2 - \nu) \frac{m^2 \pi^2}{p^2} \}] = 0. \quad (5.87b)$$

In view of the relations (5.74), Eqs. (5.87) can be written as

$$A_1 (\bar{s} \cosh \alpha + \bar{t} \cos \beta - \delta \Theta \bar{t} \sin \beta) + A_2 (\bar{s} \sinh \alpha + \frac{\alpha}{\beta} \bar{t} \sin \beta) = 0, \quad (5.88a)$$

$$A_1 (\alpha \bar{t} \sinh \alpha - \beta \bar{s} \sin \beta - \beta \delta \Theta \bar{s} \cos \beta) + A_2 (\alpha \bar{t} \cosh \alpha + \alpha \bar{s} \cos \beta) = 0, \quad (5.88b)$$

where $\Theta = (\alpha^2 + \beta^2)/\beta$. It should be noted that Eqs. (5.88a) and (5.88b) are identical to Eqs. (5.75a) and (5.75b) for $\delta = 0$ which corresponds to one end being fixed and the other end free. For a nontrivial solution, the determinant of the coefficients of A_1 and A_2 must vanish. That is,

$$\begin{vmatrix} \bar{s} \cosh \alpha + \bar{t} \cos \beta - \delta \Theta \bar{t} \sin \beta & \bar{s} \sinh \alpha + \frac{\alpha}{\beta} \bar{t} \sin \beta \\ \alpha \bar{t} \sinh \alpha - \beta \bar{s} \sin \beta - \beta \delta \Theta \bar{s} \cos \beta & \alpha \bar{t} \cosh \alpha + \alpha \bar{s} \cos \beta \end{vmatrix} = 0. \quad (5.89)$$

The characteristic equation is

$$2\bar{s}\bar{t} + (\bar{t}^2 + \bar{s}^2) \cos \beta \cosh \alpha + \frac{1}{\alpha\beta} (\beta^2 \bar{s}^2 - \alpha^2 \bar{t}^2) \sin \beta \sinh \alpha + \delta \Theta (-\bar{t}^2 \sin \beta \cosh \alpha + \bar{s}^2 \frac{\beta}{\alpha} \cos \beta \sinh \alpha) = 0. \quad (5.90)$$

Dividing by $\cos \beta \cosh \alpha$ throughout, we get

$$\frac{2\bar{s}\bar{t}}{\cos \beta \cosh \alpha} + (\bar{t}^2 + \bar{s}^2) - \frac{1}{\alpha\beta}(\alpha^2\bar{t}^2 - \beta^2\bar{s}^2) \tan \beta \tanh \alpha - \delta\Theta(\bar{t}^2 \tan \beta - \bar{s}^2 \frac{\beta}{\alpha} \tanh \alpha) = 0. \quad (5.91)$$

The critical buckling stress $(\sigma_x)_{cr}$ can now be expressed by Eq. (5.58), i.e.,

$$(\sigma_x)_{cr} = \frac{kE}{12(1 - \nu^2)} \left(\frac{t}{b}\right)^2. \quad (5.58)$$

Here, k depends on δ which in turn depends on the relative dimensions of the plate and the supporting structure.

EXAMPLE 5.5 (Buckling of plate elastically restrained along two opposite edges) To generalize the technique we shall use, we assume that the coefficient of restraint is different for the two opposite edges of the given plate. This implies that the flange dimensions are not the same. The web of an I -section is a typical example of a plate where the coefficient of restraint on the two opposite edges is the same.

The general solution to the differential equation (5.42) is

$$w(\xi, \eta) = \sin m\pi\xi(A_1 \cosh \alpha\eta + A_2 \sinh \alpha\eta + A_3 \cos \beta\eta + A_4 \sin \beta\eta), \quad (5.78)$$

where α and β are as defined by Eqs. (5.65). The constants A_1, A_2, A_3 , and A_4 are determined from the boundary conditions which are

$$w = 0 \quad (\eta = 0, 1), \quad (5.92a)$$

$$\phi = \bar{\phi} \quad (\eta = 0, 1). \quad (5.92b)$$

Following the steps given in Example 5.4, we can write the boundary conditions (5.92b) as

$$\frac{\partial w}{\partial \eta} - \delta_1 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0), \quad (5.93a)$$

$$\frac{\partial w}{\partial \eta} + \delta_2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 1), \quad (5.93b)$$

where δ_1 and δ_2 are the coefficients of restraint of $\eta = 0$ and $\eta = 1$, respectively. Substituting Eq. (5.78) in Eqs. (5.92a) and (5.93), we get

$$A_1 + A_3 = 0, \quad (5.94a)$$

$$A_1 \cosh \alpha + A_2 \sinh \alpha + A_3 \cos \beta + A_4 \sin \beta = 0, \quad (5.94b)$$

$$A_2\alpha + A_4\beta - \delta_1(A_1\alpha^2 - A_3\beta^2) = 0, \quad (5.94c)$$

$$A_1\alpha \sinh \alpha + A_2\alpha \cosh \alpha - A_3\beta \sin \beta + A_4\beta \cos \beta + \delta_2(A_1\alpha^2 \cosh \alpha + A_2\alpha^2 \sinh \alpha - A_3\beta^2 \cos \beta - A_4\beta^2 \sin \beta) = 0. \quad (5.94d)$$

Eliminating A_3 and A_4 from Eqs. (5.94b) and (5.94c), after substituting for A_3 and A_4 from (5.94a) and (5.94d), we obtain

$$A_1(\cosh \alpha - \cos \beta + \delta_1 \Theta \sin \beta) + A_2(\sinh \alpha - \frac{\alpha}{\beta} \sin \beta) = 0, \quad (5.95a)$$

$$A_1[\alpha \sinh \alpha + \beta \sin \beta + \beta \delta_1 \Theta \cos \beta + \delta_2(\alpha^2 \cosh \alpha + \beta^2 \cos \beta - \beta^2 \delta_1 \Theta \sin \beta)] \\ + A_2[\alpha \cosh \alpha - \alpha \cos \beta + \delta_2(\alpha^2 \sinh \alpha + \alpha \beta \sin \beta)] = 0. \quad (5.95b)$$

Putting the determinant of the coefficients of A_1 and A_2 equal to zero, we get the characteristic equation from which we can obtain the critical load $(N_x)_{cr}$.

5.3 ENERGY APPROACH

The equilibrium approach we discussed in Section 5.2 is useful for simple problems. On the other hand, for nonuniform plates, stiffened plates, and plates subjected to varying edge loads, this approach is not applicable. In these situations, we have therefore to look for approximate techniques which have the ability to converge to the exact solution. One such technique, as noted in Section 1.2.2, is the energy approach. An advantage in following this approach is that, if required, we can obtain not only the governing equation but also the boundary conditions for a problem.

In Section 1.2.2, we used the energy method to derive the critical load of a column by considering the strain energy due to the bending stress σ_x . A plate, being in a two-dimensional plane stress state, has the possible internal stresses σ_x , σ_y , and τ_{xy} . The strain energy stored in it is

$$U = \frac{1}{2} \int_{\text{vol}} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}) dv. \quad (5.96)$$

It can be expressed in terms of the transverse displacement w by making use of the constitutive equations and the relations between stress and curvature. Carrying out the integration with respect to z , we can write Eq. (5.96) as

$$U = \frac{1}{2} \int_0^a \int_0^b D \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy. \quad (5.97)$$

To obtain the expression for the potential energy due to the external loads, each load is separately considered, then the potential energy for each such load is estimated, and, finally, the results so obtained are added.

Potential energy due to N_x

Consider a strip dy of a plate as shown in Fig. 5.9. The load acting on this strip is $N_x dy$. Following Eq. (1.29), we can write the potential energy for the strip due to N_x as

$$dv_{e1} = -(N_x dy) \int_0^a \left(\frac{\partial w}{\partial x} \right)^2 dx. \quad (5.98)$$

The potential energy for the plate due to N_x can then be obtained by integrating Eq. (5.98) with respect to y . Thus,

$$v_{e1} = -\frac{1}{2} \int_0^b \int_0^a N_x \left(\frac{\partial w}{\partial x} \right)^2 dx dy. \quad (5.99)$$

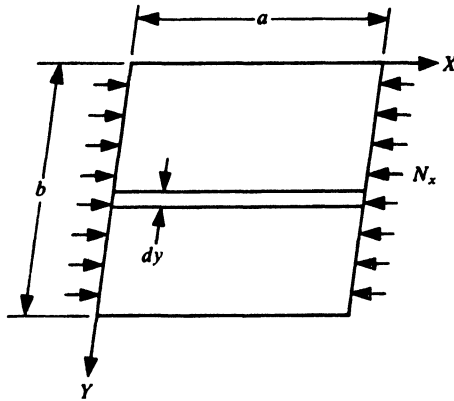


Fig. 5.9 Plate Subjected to Inplane Load.

Similarly, considering first a strip of width dx and then integrating with respect to y and x , we can write the potential energy for the plate due to N_y as

$$v_{e2} = -\frac{1}{2} \int_0^b \int_0^a N_y \left(\frac{\partial w}{\partial y} \right)^2 dx dy. \quad (5.100)$$

To determine the potential energy due to N_{xy} , we require the knowledge of shearing strain resulting from the transverse displacement w . For this, consider an element $dx dy$ of the plate (Fig. 5.9) along the X - and Y -axis as shown in Fig. 5.10. The final position of

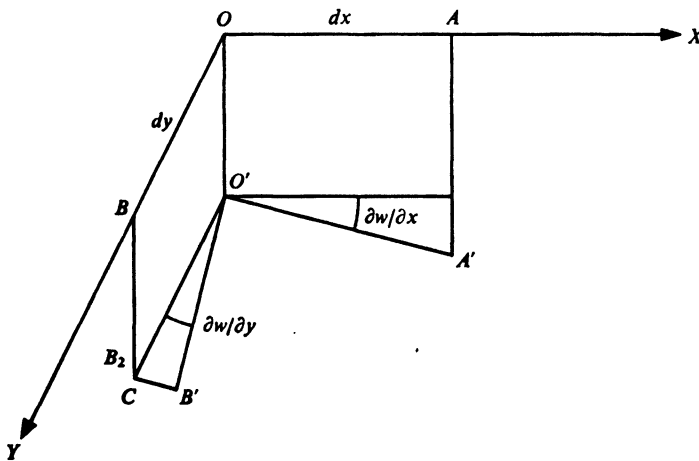


Fig. 5.10 Deformed Element Due to Shear.

this element after displacement is given by $A'O'B'$. Our interest is to find the difference

between the angles $A'O'B'$ and $\pi/2$. To do this, consider the triangle $B_2O'A'$. Rotating this angle with respect to $O'A'$ by an angle $\partial w/\partial y$, we move to a point C such that $CO'A'$ is in the same plane as $B'O'A'$. The displacement B_2C is equal to $(\partial w/\partial y) dy$ and is inclined to the vertical by an angle $\partial w/\partial x$. The angle $CO'B'$ represents the strain and is given by $(\partial w/\partial y)(\partial w/\partial x)$. Thus,

$$v_{e3} = -\frac{1}{2} \int_0^a \int_0^b 2N_{xy} \left(\frac{\partial w}{\partial y} \right) \left(\frac{\partial w}{\partial x} \right) dx dy. \quad (5.101)$$

The factor 2 inside the integral is due to the contribution from N_{yx} . Combining v_{e1} , v_{e2} , and v_{e3} , we get

$$V_e = -\frac{1}{2} \int_0^a \int_0^b \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) dx dy. \quad (5.102)$$

The total potential Π_p is then

$$\Pi_p = U + V_e$$

At buckling, the total potential must have a stationary value. Hence,

$$\delta(\Pi_p) = \delta(U + V_e) = 0. \quad (5.103)$$

5.4 APPROXIMATE TECHNIQUES

5.4.1 Rayleigh-Ritz Method

The importance of the Rayleigh-Ritz method lies in the fact that it offers a means for obtaining an approximate solution when getting a solution by the differential equation approach becomes too difficult or is not practicable. The accuracy of the solution thus obtained depends largely on the displacement function chosen. Generally, such a function can be a polynomial, a bar buckling function (i.e., the mode shape of a column with the same end conditions as the given plate), or a beam characteristic function (i.e., the mode shape of a freely vibrating beam), the main requirement being that it satisfy the geometric boundary conditions of the given plate. Further, it has been observed that the convergence of the solution is faster if an orthogonal function (e.g., beam characteristic function) is used as the displacement function. The method is essentially the same as outlined in Section 1.8.2. However, a plate being two-dimensional, the integration has to be carried out with respect to both x and y .

5.4.2 Galerkin's Technique

In a situation where it is not possible to obtain a closed-form solution through the governing differential equation, we can obtain an approximate solution by extending the Galerkin integral (1.195) which is true for a one-dimensional member. Unlike in the Rayleigh-Ritz technique, the approximating function here must satisfy both geometric and natural (dynamic) boundary conditions. This restricts the choice of the approximating function. Many a time, the result obtained by this method is better than that gotten by the Rayleigh-Ritz technique.

The total potential Π_p is given by

$$\begin{aligned} \Pi_p = \int_0^a \int_0^b & \left[\frac{D}{2} (\nabla^2 w)^2 + D(1-\nu) \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} - \frac{1}{2} N_x \left(\frac{\partial w}{\partial x} \right)^2 \right. \\ & \left. - \frac{1}{2} N_y \left(\frac{\partial w}{\partial y} \right)^2 - N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy. \end{aligned} \quad (5.104)$$

Following the steps enumerated in Section 1.8.3, if we give w a small variation δw satisfying the boundary conditions, the corresponding variation $\delta \Pi_p$ then is zero. Hence, from Eq. (5.104),

$$\begin{aligned} \int_0^a \int_0^b & [D \nabla^2 w \delta^2 w + D(1-\nu) \{ 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \delta w}{\partial x \partial y} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \delta w}{\partial y^2} - \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \delta w}{\partial x^2} \} - N_y \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \\ & - N_{xy} \{ \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} \} - N_x \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x}] dx dy = 0. \end{aligned} \quad (5.105)$$

After integrating Eq. (5.105) by parts, we arrive at the boundary conditions and find that

$$\begin{aligned} \int_0^a \int_0^b & [\nabla^2 (D \nabla^2 w) + (1-\nu) \{ 2 \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \} + N_x \frac{\partial^2 w}{\partial x^2} \\ & + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y}] \delta w dx dy = 0. \end{aligned} \quad (5.106)$$

Equation (5.106) is called the Galerkin equation.

In what follows, we shall demonstrate the application of the Rayleigh-Ritz and Galerkin techniques first to buckling of uniform plates and then to buckling of nonuniform plates.

EXAMPLE 5.6 (Buckling load of rectangular plate axially compressed, simply-supported along two opposite edges and clamped along other two edges) The plate shown in Fig. 5.11 has sides of length a and b and is compressed by a uniformly distributed load N_x

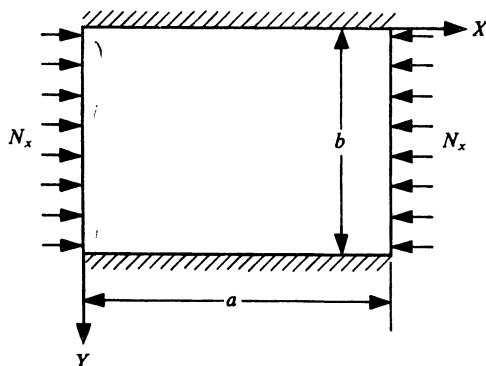


Fig. 5.11 Rectangular Plate under Compressive Loading.

acting along two opposite, simply-supported edges. The boundary conditions in terms of the nondimensional parameters ξ and η are

$$w = \frac{\partial w}{\partial \xi} = 0 \quad (\xi = 0, 1), \quad (5.107a)$$

$$w = \frac{\partial^2 w}{\partial \xi^2} + \nu p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0, 1), \quad (5.107b)$$

where $p (=a/b)$ is the aspect ratio of the plate. A one-term solution of the type

$$w(\xi, \eta) = A \sin m\pi\xi(1 - \cos 2\pi\eta) \quad (5.108)$$

satisfies the boundary conditions exactly. The reason we have chosen this shape function, as shown in Example 5.1, is that at the minimum critical load, a plate, irrespective of its size, buckles with one half wave along the Y -direction. The nondimensional form of the strain energy U is

$$U = \frac{1}{2} \frac{D}{pa^2} \int_0^1 \int_0^1 \left\{ \left(\frac{\partial^2 w}{\partial \xi^2} + p^2 \frac{\partial^2 w}{\partial \eta^2} \right)^2 - 2(1 - \nu) \left[p^2 \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - p^2 \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right)^2 \right] \right\} d\xi d\eta. \quad (5.109) \quad \checkmark$$

To evaluate this, we need the following derivatives of w :

$$\begin{aligned} \frac{\partial w}{\partial \xi} &= Am\pi \cos m\pi\xi(1 - \cos 2\pi\eta), \\ \frac{\partial w}{\partial \eta} &= A2\pi \sin m\pi\xi \sin 2\pi\eta, \\ \frac{\partial^2 w}{\partial \xi^2} &= -Am^2\pi^2 \sin m\pi\xi(1 - \cos 2\pi\eta), \\ \frac{\partial^2 w}{\partial \eta^2} &= A4\pi^2 \sin m\pi\xi \cos 2\pi\eta, \\ \frac{\partial^2 w}{\partial \xi \partial \eta} &= A4m\pi^2 \cos m\pi\xi \sin 2\pi\eta. \end{aligned} \quad (5.110)$$

Substituting derivatives (5.110) in Eq. (5.109), we get

$$\begin{aligned} U &= \frac{D}{2} \frac{1}{pa^2} \int_0^1 \int_0^1 A^2 \{ [m^4\pi^4 \sin^2 m\pi\xi(1 - \cos 2\pi\eta)^2 \\ &\quad + 16p^4\pi^4 \sin^2 m\pi\xi \cos^2 2\pi\eta - 8m^2p^2\pi^4 \sin^2 m\pi\xi \cos 2\pi\eta(1 - \cos 2\pi\eta)] \\ &\quad + 2(1 - \nu)[p^2 4m^2\pi^4 \sin^2 m\pi\xi(1 - \cos 2\pi\eta) \cos 2\pi\eta \\ &\quad + p^2 4m^2\pi^4 \cos^2 m\pi\xi \sin^2 2\pi\eta] \} d\xi d\eta. \end{aligned} \quad (5.111)$$

Hence,

$$U = \frac{D}{2} \frac{A^2\pi^4}{pa^2} \frac{1}{2} (3m^4 + 4p^4 + 8m^2p^2). \quad (5.112)$$

The nondimensional form of the potential energy V_e is

$$\begin{aligned}
 V_e &= -\frac{1}{2} \int_0^1 \int_0^1 \frac{N_x}{p} \left(\frac{\partial w}{\partial \xi} \right)^2 d\xi d\eta \\
 &= -\frac{1}{2} \int_0^1 \int_0^1 \frac{N_x}{p} A^2 m^2 \pi^2 \cos^2 m\pi\xi (1 - \cos 2\pi\eta)^2 d\xi d\eta \\
 &= -\frac{1}{2} \frac{N_x}{p} A^2 m^2 \pi^2 \frac{3}{4}.
 \end{aligned} \tag{5.113}$$

The total potential for the plate is given by

$$\Pi_p = U + V_e = \frac{D}{2} \frac{A^2 \pi^4}{4p a^2} (3m^4 + 4p^4 + 8m^2 p^2) - \frac{1}{2} \frac{N_x}{p} \frac{3}{4} A^2 m^2 \pi^2. \tag{5.114}$$

Taking the variation of Π_p with respect to A and equating it to zero, we get

$$A \left[\frac{D \pi^4}{4p a^2} (3m^4 + 4p^4 + 8m^2 p^2) - \frac{N_x}{p} \frac{3}{4} m^2 \pi^2 \right] = 0. \tag{5.115}$$

For a nontrivial solution, $A \neq 0$; hence, the quantity within the square brackets must be zero. Replacing N_x by $(N_x)_{cr}$, we obtain

$$(N_x)_{cr} = \frac{1}{2} \frac{D \pi^2}{m^2 a^2} (3m^4 + 16p^4 + 8m^2 p^2). \tag{5.116}$$

For a square plate ($p = 1$) with $m = 1$,

$$(N_x)_{cr} = 9D\pi^2/a^2. \tag{5.117}$$

This is an upper bound to the exact solution and is in conformity with the observations we made in Section 1.8.1. For other values of the aspect ratio (p), we can similarly obtain the critical loads for different values of m and then pick up the one which is minimum.

EXAMPLE 5.7 (Buckling load of clamped rectangular plate uniformly compressed along one direction) Consider the rectangular plate with all sides clamped shown in Fig. 5.11. Here, since it is not possible to analyze the plate through the differential equation approach, we shall attempt a solution, using the energy approach.

The boundary conditions are

$$w = \partial w / \partial \xi = 0 \quad (\xi = 0, 1), \tag{5.118a}$$

$$w = \partial w / \partial \eta = 0 \quad (\eta = 0, 1). \tag{5.118b}$$

These can be satisfied if we assume w to be of the form

$$w = A(1 - \cos 2\pi\xi)(1 - \cos 2\pi\eta). \tag{5.119}$$

This is a one-term representation for the displacement. An improved solution can be obtained by taking two or three terms in the representation for displacement.

To evaluate U and V_e , we need the following derivatives of w :

$$\begin{aligned}\frac{\partial w}{\partial \xi} &= 2\pi A \sin 2\pi\xi(1 - \cos 2\pi\eta), \\ \frac{\partial^2 w}{\partial \xi^2} &= 4\pi^2 A \cos 2\pi\xi(1 - \cos 2\pi\eta), \\ \frac{\partial w}{\partial \eta} &= 2\pi A(1 - \cos 2\pi\xi) \sin 2\pi\eta, \\ \frac{\partial^2 w}{\partial \eta^2} &= 4\pi^2 A(1 - \cos 2\pi\xi) \cos 2\pi\eta.\end{aligned}\tag{5.120}$$

Substituting derivatives (5.120) in the expressions for U and V_e , i.e., (5.109) and (5.113), and integrating, we get

$$U = \frac{D}{2pa^2} \frac{16\pi^4 A^2}{4} (3 + 3p^4 + 2p^2),\tag{5.121}$$

$$V_e = -\frac{1}{2} \frac{N_x}{p} 4\pi^4 A^2 \frac{1}{4}.\tag{5.122}$$

The total potential Π_p can now be written as

$$\Pi_p = \frac{2D\pi^4 A^2}{pa^2} (3 + 3p^4 + 2p^2) - \frac{1}{2} \frac{N_x}{p} \pi^2 A^2.\tag{5.123}$$

Taking the variation of Π_p with respect to A and equating it to zero, we get

$$(N_x)_{cr} = \frac{4}{3}\pi^2 D(3 + 3p^4 + 2p^2).\tag{5.124}$$

For a square plate ($p = 1$), this reduces to

$$(N_x)_{cr} = 10.67\pi^2 D/a^2.\tag{5.125}$$

Using an infinite series for w , Levy [7] obtained the exact solution, namely,

$$(N_x)_{cr} = (10.07) \frac{\pi^2 D}{a^2},\tag{5.126}$$

of the example we have discussed.

As can be noted, the energy technique gives an upper bound to the exact value. Further, the error in the one-term solution (5.119) is of the order of 6 per cent. This can be minimized if we use a two-term representation as

$$w = A(1 - \cos 2\pi\xi)(1 - \cos 2\pi\eta) + B(1 - \cos 4\pi\xi)(1 - \cos 2\pi\eta)\tag{5.127}$$

which is based on the observation that, in general, at the critical load, a plate buckles with one half wave along the Y -direction.

EXAMPLE 5.8 (Buckling of simply-supported rectangular plate compressed in two perpendicular directions) Consider a rectangular plate simply-supported on all sides and

subjected to uniformly distributed compressive forces N_x and N_y (see Fig. 5.12). The

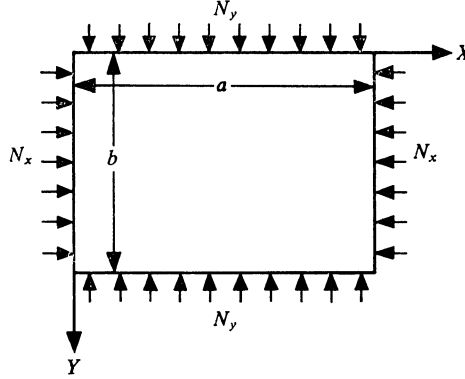


Fig. 5.12 Simply-Supported Rectangular Plate Compressed in Two Perpendicular Directions.

boundary conditions are

$$w = \frac{\partial^2 w}{\partial \xi^2} + \nu p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\xi = 0, 1), \quad (5.128a)$$

$$w = p^2 \frac{\partial^2 w}{\partial \eta^2} + \nu \frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\eta = 0, 1). \quad (5.128b)$$

Since $w = 0$ everywhere along the edges, the moment conditions reduce to

$$\partial^2 w / \partial \xi^2 = 0 \quad (\xi = 0, 1), \quad (5.129a)$$

$$\partial^2 w / \partial \eta^2 = 0 \quad (\eta = 0, 1). \quad (5.129b)$$

An expression of the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi\xi \sin n\pi\eta \quad (5.130)$$

satisfies the boundary conditions exactly. Substituting expression (5.130) in Eqs. (5.109) and (5.113), we get, on integration,

$$U = \frac{1}{8} \frac{D\pi^4}{pa^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 (m^2 + p^2 n^2)^2, \quad (5.131)$$

$$V_c = -\frac{\pi^2}{8p} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn}^2 (N_x m^2 + p^2 N_y). \quad (5.132)$$

To find the critical load, we shall consider only one term in the double series for w , i.e., (5.130), and then calculate the critical values of N_x and N_y . For this, taking the variation

of the total potential Π_p , we get

$$(N_x m^2 + p^2 n^2 N_y) = \frac{D\pi^2}{a^2} (m^2 + p^2 n^2)^2. \quad (5.133)$$

Introducing the notation

$$D\pi^2/a^2 = N_e, \quad (5.134)$$

we obtain

$$N_x m^2 + p^2 n^2 N_y = N_e (m^2 + p^2 n^2)^2. \quad (5.135)$$

Equation (5.135) is referred to as the interaction equation. For a given aspect ratio of the plate, it tells us the ratio of N_x and N_y for the plate to buckle. For instance, consider a square plate of sides a . Equation (5.135) then reduces to

$$N_x m^2 + n^2 N_y = N_e (m^2 + n^2)^2. \quad (5.136)$$

If only N_x is applied, we get

$$N_x m^2 = N_e (m^2 + n^2)^2. \quad (5.137)$$

In Example 5.1, we saw that the minimum critical load is obtained when $m = n = 1$. Hence,

$$N_x = 4N_e = 4\pi^2 D/a^2. \quad (5.138)$$

This result is the same as that obtained in Example 5.1. With N_x being zero, we have

$$N_y = 4\pi^2 D/a^2. \quad (5.139)$$

If both N_x and N_y are present such that $N_x = N_y = N$, we obtain

$$N = 2\pi^2 D/a^2 = \frac{1}{2}N_y = \frac{1}{2}N_x, \quad (5.140)$$

that is, the critical load then is just half that for a square plate compressed in only one direction. The interested reader may see Timoshenko and Gere [6] for more details on buckling under biaxial loading.

EXAMPLE 5.9 (Buckling of simply-supported rectangular plate subjected to pure shear) Consider a rectangular plate simply-supported on all sides and subjected to pure shear (see Fig. 5.13). A typical example of such a plate is the thin web of an *I*-section girder.

A close look at the governing equation (5.42) reveals that it is not practicable to obtain the solution by using the differential equation approach because of the presence of cross derivative terms in it. In this situation therefore, the only technique we can use is the energy approach, and thus obtain a closed-form solution for w in the form of a series. The boundary conditions at the edges can be satisfied by taking a solution for w as

$$w = \sum_{m=1}^q \sum_{n=1}^q A_{mn} \sin m\pi\xi \sin n\pi\eta.$$

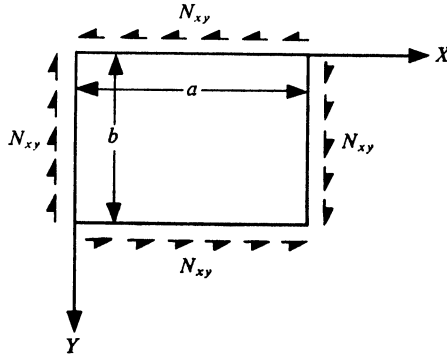


Fig. 5.13 Simply-Supported Rectangular Plate Subjected to Pure Shear.

The expression for U here remains the same as in Example 5.8 and is given by Eq. (5.131), i.e.,

$$U = \frac{1}{8} \frac{D\pi^4}{pa^2} \sum_{m=1}^q \sum_{n=1}^q A_{mn}^2 (m^2 + p^2 n^2)^2.$$

The potential energy V_e has to be obtained from the expression

$$V_e = - \int_0^1 \int_0^1 N_{xy} \frac{\partial w}{\partial \eta} \frac{\partial w}{\partial \xi} d\xi d\eta. \quad (5.141)$$

Substituting for $\partial w/\partial \xi$ and $\partial w/\partial \eta$, we get

$$V_e = - \int_0^1 \int_0^1 N_{xy} \sum_{m=1}^q \sum_{n=1}^q \sum_{r=1}^q \sum_{s=1}^q (A_{mn} A_{rs} \cos m\pi\xi \sin n\pi\eta \sin r\pi\xi \cos s\pi\eta) m s \pi^2.$$

Observing that

$$\begin{aligned} \int_0^1 \sin m\pi\xi \cos r\pi\xi &= 0 & \text{if } m \pm r \text{ is an even number,} \\ \int_0^1 \sin m\pi\xi \cos r\pi\xi &= \frac{2}{\pi} \frac{m}{m^2 - r^2} & \text{if } m \pm r \text{ is an odd number,} \end{aligned}$$

we find that the expression for potential energy simplifies to

$$V_e = -4N_{xy} \sum_m^q \sum_n^q \sum_r^q \sum_s^q A_{mn} A_{rs} \frac{m n r s}{(m^2 - r^2)(s^2 - n^2)}. \quad (5.142)$$

Hence,

$$\Pi_p = \frac{1}{8} \frac{D\pi^4}{pa^2} \sum_{m=1}^q \sum_{n=1}^q A_{mn}^2 (m^2 + p^2 n^2)^2 - 4N_{xy} \sum_m^q \sum_n^q \sum_r^q \sum_s^q A_{mn} A_{rs} \frac{m n r s}{(m^2 - r^2)(s^2 - n^2)}. \quad (5.143)$$

We are interested in finding those values of m, n which make N_{xy} a minimum. This can be done by taking the derivative of N_{xy} with respect to each of the coefficients A_{mn} and equating it to zero. The system of equations that thus results can be divided into two sets of equations, one for which $(m + n)$ are odd numbers, and the other for which $(m + n)$ are even numbers. For the sake of discussion, we shall restrict the number of terms in the series representation for w by assuming that $q = 2$. Then, m, n, r , and s can each have the values 1 and 2 as shown:

m	n	r	s
1	1	2	2
1	2	2	1
2	1	1	2
2	2	1	1

Consequently, Eq. (5.142) will have four terms because only the odd combination will contribute. Substituting the various values of m and n , we find the expression for the total potential reduces to

$$U + V_e = \frac{1}{8} \frac{D\pi^4}{pa^2} [A_{11}^2(1 + p^2)^2 + A_{12}^2(1 + 4p^2)^2 + A_{21}^2(4 + p^2)^2 + A_{22}^2(4 + 4p^2)^2] \\ + \frac{32}{9} N_{xy}(A_{11}A_{22} - A_{21}A_{12}). \quad (5.144)$$

Taking the derivatives of A_{11} , A_{12} , A_{21} , and A_{22} , the set of homogeneous algebraic equations we get is

$$\frac{1}{4} \frac{D\pi^4}{pa^2} A_{11}(1 + p^2)^2 + \frac{32}{9} N_{xy} A_{22} = 0, \\ \frac{1}{4} \frac{D\pi^4}{pa^2} A_{12}(1 + 4p^2)^2 - \frac{32}{9} N_{xy} A_{21} = 0, \\ \frac{1}{4} \frac{D\pi^4}{pa^2} A_{21}(4 + p^2)^2 - \frac{32}{9} N_{xy} A_{12} = 0, \\ \frac{1}{4} \frac{D\pi^4}{pa^2} A_{22}(4 + 4p^2)^2 + \frac{32}{9} N_{xy} A_{11} = 0. \quad (5.145)$$

For a nontrivial solution, the determinant of the coefficients of A_{11} , A_{12} , A_{21} , and A_{22} must vanish. This leads to the characteristic equation. Defining

$$\delta = \frac{\pi^4}{32} \frac{D}{pa^2},$$

we can write the roots of the characteristic equation as

$$N_{1,2} = \pm \frac{8}{4} \delta (1 + p^2)(4 + 4p^2), \quad (5.146)$$

$$N_{3,4} = \pm \frac{8}{3} \delta (1 + 4p^2)(4 + p^2). \quad (5.147)$$

By assigning different values to p , we see that $N_{1,2}$ is lower than $N_{3,4}$, and hence $(N_{xy})_{cr}$ is given by N_1 or N_2 . This implies that the buckling of the plate does not depend on the sense of shear stress. Thus,

$$(N_{xy})_{cr} = \frac{8}{3} \delta (1 + p^2)(4 + 4p^2). \quad (5.148)$$

Substituting for δ , we get

$$(N_{xy})_{cr} = \frac{9}{32} \frac{Et^3 \pi^4}{12(1 - \nu^2) p a^2} (1 + p^2)^2, \quad (5.149)$$

and therefore

$$(\tau_{xy})_{cr} = \frac{\pi^2 E}{12(1 - \nu^2)} k \left(\frac{t}{b}\right)^2, \quad (5.150)$$

where

$$k = \frac{9\pi^2}{32} \frac{(1 + p^2)^2}{p^3}. \quad (5.151)$$

For a square plate ($p = 1$), we obtain

$$(\tau_{xy})_{cr} = \frac{\pi^2 E}{2(1 - \nu^2)} \frac{9\pi^2}{8} \left(\frac{t}{b}\right)^2. \quad (5.152)$$

Timoshenko and Gere [6] have shown that this value is in error by +15 per cent. Further, the value of k for a square plate as given by them is 9.4 by taking five terms in the series for displacement w .

EXAMPLE 5.10 (Buckling of simply-supported rectangular plate subjected to varying inplane axial load along two opposite edges) In the examples we have so far given, we assumed that the axial load acting on the plate is of uniform intensity. In practice, such a situation does not exist if the plate is thick and is capable of carrying bending stresses.

Consider a plate (Fig. 5.14) subjected to an inplane load varying with respect to y . The magnitude of the load at a distance η from the upper edge of the plate is given by

$$N_x = N_0(1 - \alpha\eta). \quad (5.153)$$

$\alpha = 0$ corresponds to a uniformly distributed compressive load, $\alpha = 2$ to pure bending, and $0 < \alpha < 2$ to

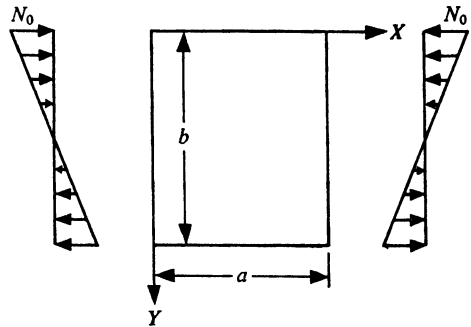


Fig. 5.14 Rectangular Plate Subjected to Variable Edge Load.

combination of bending and compression. Here again, the solution by the differential equation approach is complicated, if not impossible, since the governing equation (5.42), on substitution for N_x , becomes a variable coefficient type of differential equation. The energy approach, as already noted, gives an upper bound to the exact solution. We shall therefore use this technique to obtain an approximate solution.

In Example 5.1, we saw that a simply-supported plate under axial compression buckles into equal half waves with the nodal lines perpendicular to the X -axis. Accordingly, we shall take each buckle as representing a plate simply-supported on all its four edges.

Assume that the deflection of a buckled plate simply-supported on all sides is

$$w = \sin \pi \eta \sum_{n=1}^m A_n \sin n \pi \xi. \quad (5.154)$$

The strain energy U is obtained from Eq. (5.109). Substituting for w from Eq. (5.154) and integrating, we get

$$U = \frac{D\pi^4}{8pa^2} \sum_{n=1}^m A_n^2 (1 + n^2 p^2)^2. \quad (5.155)$$

The potential energy V_e due to the applied load is

$$V_e = -\frac{1}{2} \frac{N_0}{p} \int_0^1 \int_0^1 (1 - \alpha \eta) \left(\frac{\partial w}{\partial \xi} \right)^2 d\xi d\eta. \quad (5.156)$$

Substituting for $\partial w / \partial \xi$ in Eq. (5.156), we get

$$V_e = -\frac{1}{2} \frac{N_0}{p} \int_0^1 \int_0^1 (1 - \alpha \eta) \pi^2 \cos^2 \pi \xi \sum_{q=1}^m \sum_{n=1}^m A_n A_q \sin n \pi \eta \sin q \pi \eta d\xi d\eta. \quad (5.157)$$

Making use of the integral relations

$$\begin{aligned} \int_0^1 \eta \sin q \pi \eta \sin n \pi \eta d\eta &= \frac{1}{2} && \text{for } n = q, \\ \int_0^1 \eta \sin q \pi \eta \sin n \pi \eta d\eta &= 0 && \text{for } n \neq q \text{ and } n \pm q \text{ is an even number,} \\ \int_0^1 \eta \sin q \pi \eta \sin n \pi \eta d\eta &= -\frac{4}{\pi^2} \frac{nq}{(q^2 - n^2)^2} && \text{for } n \neq q \text{ and } n \pm q \text{ is an odd number,} \end{aligned}$$

we obtain

$$V_e = -\frac{1}{8} \frac{N_0 \pi^2}{p} \sum_{n=1}^m A_n^2 + \alpha \frac{N_0 \pi^2}{2p} \sum_{n=1}^m A_n^2 - \frac{2N_0}{p} \sum_{n=1}^m \sum_{q=1}^m \frac{A_n A_q nq}{(q^2 - n^2)^2}. \quad (5.158)$$

The reason the factor 2 appears in the last term on the right-hand side of this equation is that n assumes all values from 1 to m , whereas q takes only such values for which $(n + q)$ is an odd number. The total potential Π_p can then be written, by combining Eqs. (5.155)

and (5.158), as

$$\begin{aligned} \Pi_p = U + V_e = & \frac{D\pi^4}{8pa^2} \sum_{n=1}^m A_n^2 (1 + n^2 p^2)^2 - \frac{1}{8} \frac{N_0 \pi^2}{p} \sum_{n=1}^m A_n^2 \\ & + \alpha_0 \frac{N}{16p} \pi^2 \sum_{n=1}^m A_n^2 - \frac{2N_0 \alpha}{p} \sum_{n=1}^m \sum_{q=1}^m \frac{A_n A_q nq}{(q^2 - n^2)^2}. \end{aligned} \quad (5.159)$$

Taking the derivative of Π_p with respect to $A_n (n = 1, 2, \dots, m)$, the system of simultaneous homogeneous algebraic equations we get is

$$\begin{aligned} & \left[\frac{2D\pi^4}{8pa^2} (1 + n^2 p^2)^2 - \frac{1}{8} \frac{N_0 \pi^2}{p} + 2\alpha \frac{N_0}{16p} \pi^2 \right] A_n \\ & - \frac{2N_0 \alpha}{p} \sum_{q=1}^m \frac{qn A_q}{(q^2 - n^2)^2} = 0 \quad (n = 1, 2, \dots, m). \end{aligned} \quad (5.160)$$

We can now calculate the critical load by successive approximation, considering different values of n . Such a method permits us to check the convergence rate of the solution.

One-term solution ($n = 1$) Substituting $n = 1$ in Eqs. (5.160), we get

$$\left[\frac{1}{4} \frac{D\pi^4}{pa^2} (1 + p^2)^2 - \frac{1}{4} \frac{N_0 \pi^2}{p} + \frac{1}{8} \frac{N_0}{p} \alpha \pi^2 \right] A_1 = 0. \quad (5.161)$$

Since $A_1 \neq 0$, the quantity within the square brackets must be zero. This leads to

$$(N_0)_{cr} = \frac{D\pi^2}{a^2} \frac{(1 + p^2)^2}{1 - \alpha/2}. \quad (5.162)$$

Substituting for p and noting that $N_0 = \sigma t$, we get

$$\sigma_{cr} = \frac{D\pi^2}{b^2 t} \frac{(a/b + b/a)^2}{1 - \alpha/2}. \quad (5.163)$$

For $\alpha = 0$, i.e., uniform compression, the expression for σ_{cr} coincides with that for uniform compressive stress (see Example 5.1). This solution, however, will not be accurate for large α .

Two-term solution ($n = 1, 2$) To obtain a better approximation, we consider two terms for n . Writing Eqs. (5.160) for $n = 1, 2$, we have

$$\left[\frac{D\pi^4}{a^2} (1 + p^2)^2 + \left(\frac{\alpha}{2} - 1 \right) N_0 \pi^2 \right] A_1 - \frac{1}{8} N_0 \alpha A_2 = 0, \quad (5.164a)$$

$$- \frac{1}{8} N_0 \alpha A_1 + \left[\frac{D\pi^4}{a^2} (1 + 4p^2)^2 + \left(\frac{\alpha}{2} - 1 \right) N_0 \pi^2 \right] A_2 = 0. \quad (5.164b)$$

Equating to zero the determinant of the coefficients of A_1 and A_2 , we obtain

$$\left(\frac{D\pi^4}{a^2}\right)^2(1+p^2)^2(1+4p^2)^2 + \frac{D\pi^4}{a^2}\left(\frac{\alpha}{2}-1\right)N_0\pi^2[(1+4p^2)^2+(1+p^2)^2] \\ + \left(\frac{\alpha}{2}-1\right)^2N_0^2\pi^4 - \frac{256}{81}N_0^2\alpha^2 = 0. \quad (5.165)$$

For a square plate ($p = 1$) with $\alpha = 2$ (pure bending), Eq. (5.165) reduces to

$$N_0 = \frac{D\pi^4}{a^2} \frac{48}{16}. \quad (5.166)$$

Hence,

$$\sigma_{cr} = k \frac{\pi^2 E}{12(1-\nu^2)} \left(\frac{t}{b}\right)^2, \quad (5.167)$$

where

$$k = 45\pi^2/16 = 27.333.$$

This differs from the exact value ($k = 25.6$) by about 8 per cent. Timoshenko and Gere [6] have shown that exact value is obtained by taking $n = 4$ in the series representation for the displacement w .

In Examples 5.1–5.10, we considered rectangular plates of uniform thickness. In a structure such as an aircraft and a space vehicle, where weight saving is important, it is advisable that all sections be subjected to maximum stress levels. This can be achieved only when the cross-section of each such section is nonuniform. As an example, consider a uniform cantilever beam carrying a tip load. Here, the bending moment varies linearly from zero to a maximum at the root. Thus, if the beam is of uniform cross-section, the stress varies from zero at the tip to a maximum at the root, thereby implying that all the sections of the beam are not subjected to the same stress level. Therefore, for a better utilization of the material, it is desirable that the sections be tapered. In a two-dimensional structure, e.g., plate, depending on the design requirements, the thickness can vary linearly or nonlinearly along the X - and Y -direction.

EXAMPLE 5.11 (Buckling of uniformly tapered rectangular plate subjected to uniaxial compressive load) Consider a rectangular plate simply-supported on all sides. Assume that the thickness of the plate along the X -axis varies as $\beta(x/a)^{1/3}$ and along the Y -axis it is constant. The governing equation for a general nonuniform plate with all the possible inplane loads is given by (5.32). Particularizing (5.32) to the structure we are considering, we find this governing equation becomes

$$\frac{E\beta^3}{12(1-\nu^2)}\left(\frac{x}{a}\right)\nabla^4 w + \frac{2E\beta^3}{12(1-\nu^2)}\frac{1}{a}\left(\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2}\right) \\ + (1-\nu)\frac{E\beta^3}{12(1-\nu^2)}\frac{1}{a}\frac{\partial^3 w}{\partial x \partial y^2} + N_x\frac{\partial^2 w}{\partial x^2} = 0. \quad (5.168)$$

Defining

$$D_0 = E\beta^3/[12(1-\nu^2)],$$

we can write Eq. (5.168) as

$$x\nabla^4 w + 2\left(\frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial x \partial y^2}\right) + (1 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} + a \frac{N_x}{D_0} \frac{\partial^2 w}{\partial x^2} = 0. \quad (5.169)$$

Equation (5.169) is a variable coefficient partial differential equation, and hence it is rather difficult, if not impossible, to obtain its closed-form solution. At this stage, we would like to point out that the situation we are considering is approximate since, in practice, at $x = 0$, a plate always has some finite thickness.

For our plate problem, we shall obtain the approximate buckling load by the Rayleigh-Ritz technique. The strain energy U in the nondimensional form is

$$U = \frac{1}{2pa^2} \int_0^1 \int_0^1 D \left\{ \left(\frac{\partial^2 w}{\partial \xi^2} + p^2 \frac{\partial^2 w}{\partial \eta^2} \right)^2 - 2(1 - \nu) \left[p^2 \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - p^2 \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right)^2 \right] \right\} d\xi d\eta. \quad (5.109)$$

Defining

$$D = \frac{E\beta^3}{12(1 - \nu^2)} \xi = D_0 \xi, \quad (5.170)$$

we can write U as

$$U = \frac{D_0}{2pa^2} \int_0^1 \int_0^1 \xi \left\{ \left(\frac{\partial^2 w}{\partial \xi^2} + p^2 \frac{\partial^2 w}{\partial \eta^2} \right)^2 - 2(1 - \nu) \left[p^2 \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - p^2 \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right)^2 \right] \right\} d\xi d\eta. \quad (5.171)$$

A one-term solution of the type

$$w(\xi, \eta) = A \sin m\pi\xi \sin \pi\eta \quad (5.172)$$

satisfies the boundary conditions exactly. We shall consider only one half mode since (see Example 5.1), at the critical load, the mode shape of a plate, irrespective of the plate aspect ratio, along the Y -direction is only one half wave. (However, as we have already noted, the mode shape along the X -axis can, depending on the plate aspect ratio, be several half waves.) To evaluate U , we need the following derivatives of w :

$$\begin{aligned} \frac{\partial w}{\partial \xi} &= Am\pi \cos m\pi\xi \sin \pi\eta, \\ \frac{\partial w}{\partial \eta} &= A\pi \sin m\pi\xi \cos \pi\eta, \\ \frac{\partial^2 w}{\partial \xi^2} &= -Am^2\pi^2 \sin m\pi\xi \sin \pi\eta, \\ \frac{\partial^2 w}{\partial \eta^2} &= -A\pi^2 \sin m\pi\xi \sin \pi\eta, \\ \frac{\partial^2 w}{\partial \xi \partial \eta} &= Am\pi^2 \cos m\pi\xi \cos \pi\eta. \end{aligned} \quad (5.173)$$

Substituting relations (5.173) in Eq. (5.171) and integrating, we get

$$U = \frac{D_0}{2pa^2} \frac{A^2\pi^4}{4} (m^2 + p^2)^2. \quad (5.174)$$

The potential energy V_e is

$$\begin{aligned}
 V_e &= -\frac{1}{2} \int_0^1 \int_0^1 \frac{N_x}{p} \left(\frac{\partial w}{\partial \xi} \right)^2 d\xi d\eta \\
 &= -\frac{1}{2} \frac{N_x}{p} \int_0^1 \int_0^1 A^2 m^2 \pi^2 \cos^2 m\pi\xi \sin^2 \pi\eta d\xi d\eta \\
 &= -\frac{1}{2} \frac{N_x}{p} A^2 m^2 \pi^2 \frac{1}{4}.
 \end{aligned} \tag{5.175}$$

The total potential is then

$$\Pi_p = U + V_e = \frac{D_0}{8pa^2} A^2 \pi^4 (m^2 + p^2)^2 - \frac{N_x}{8p} A^2 m^2 \pi^2. \tag{5.176}$$

Taking the variation of Π_p with respect to A and equating it to zero, we get

$$A \left[\frac{D_0 \pi^4}{pa^2} (m^2 + p^2)^2 - \frac{N_x}{p} m^2 \pi^2 \right] = 0. \tag{5.177}$$

For a nontrivial solution, $A \neq 0$. Hence, the quantity within the square brackets must be zero. Replacing N_x by $(N_x)_{cr}$, we have

$$(N_x)_{cr} = \frac{D_0 \pi^2}{a^2} \frac{(m^2 + p^2)^2}{m^2}. \tag{5.178}$$

For a square plate ($p = 1$) with $m = 1$,

$$(N_x)_{cr} = 4D_0 \frac{\pi^2}{a^2}. \tag{5.179}$$

EXAMPLE 5.12 (Buckling of tapered rectangular plate uniformly compressed along axial direction) Consider a rectangular plate with sides $x = 0, a$ and $y = 0, b$ and tapered only along the X -direction. Unlike in Example 5.11, we shall assume that the plate has a finite thickness at $x = 0$. Our analysis here is based on the Galerkin equation (5.106). The interested reader may also see Wittrick and Ellen [8] for the results they obtained for a plate with exponentially varying thickness and various boundary conditions.

The thickness t of the plate is given by

$$t = t_0 \left(1 + \alpha \frac{x}{a} \right)^{1/3}. \tag{5.180}$$

The differential equation for the plate is

$$D \nabla^4 w + 2 \frac{dD}{dx} \frac{\partial}{\partial x} (\nabla^2 w) + \frac{d^2 D}{dx^2} \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} = 0. \tag{5.181}$$

A solution of the form

$$w(x, y) = f(x) \sin \frac{n\pi y}{b}$$

satisfies exactly the simply-supported boundary conditions along $y = 0, b$. On the basis of the known behaviour of uniform plates (see Example 5.1), we can obtain the lowest buckling load when $n = 1$. Thus,

$$w = W(\xi) \sin \pi \eta, \quad (5.182)$$

where

$$\xi = x/a, \quad \eta = y/b. \quad (5.183)$$

The function $W(\xi)$ can be expressed as the series

$$W(\xi) = \sum_{m=1}^{\infty} A_m F_m(\xi), \quad (5.184)$$

where each of the functions $F_m(\xi)$ satisfies the boundary conditions at $\xi = 0, 1$. The virtual displacement δW is taken in the form

$$\delta W = F_r(\xi) \sin \pi \eta. \quad (5.185)$$

For all edges simply-supported, the boundary conditions require that

$$F_m(0) = F_m'(0) = F_m(1) = F_m'(1) = 0.$$

Hence,

$$F_m(\xi) = \sin m\pi\xi. \quad (5.186)$$

Substituting Eqs. (5.182) to (5.186) in Eq. (5.106), we get

$$\begin{aligned} & \int_0^1 \int_0^1 \{ D_0 [A_1 \pi^4 (\frac{1}{a^2} + \frac{1}{b^2})^2 (1 + \alpha\xi) \sin \pi\xi \sin \pi\eta \\ & + A_2 \pi^4 (\frac{4}{a^2} + \frac{1}{b^2})^2 (1 + \alpha\xi) \sin 2\pi\xi \sin \pi\eta - 2A_1 \frac{\pi^3 \alpha}{a^2} (\frac{1}{a^2} + \frac{1}{b^2}) \cos \pi\xi \sin \pi\eta \\ & - \frac{4A_2 \pi^3 \alpha}{a^2} (\frac{4}{a^2} + \frac{1}{b^2}) \cos 2\pi\xi \sin \pi\eta] \\ & - N_x (\frac{A_1 \pi^2}{a^2} \sin \pi\eta + \frac{4A_2 \pi^2}{a^2} \sin 2\pi\xi \sin \pi\eta) \} \delta W \, d\xi \, d\eta = 0. \end{aligned} \quad (5.187)$$

Now, substituting the expression for δW for $r = 1, 2$ and integrating, we get

$$\begin{aligned} & A_1 [D_0 \frac{\pi^2}{a^2} (1 + p^2)^2 (1 + \frac{\alpha}{2}) - N_x] + A_2 \frac{16\alpha D_0}{3a^2} (4 + p^2) = 0, \\ & A_1 [-\frac{8D_0\alpha}{3a^2} (1 + p^2)] + A_2 [\frac{D_0\pi^2}{4a^2} (4 + p^2)^2 (1 + \frac{\alpha}{2}) - N_x] = 0. \end{aligned} \quad (5.188)$$

For a nontrivial solution, the determinant of the coefficients of A_1 and A_2 must be zero. This leads to the characteristic equation

$$\begin{aligned} & N_x^2 - N_x \{ \frac{D_0\pi^2}{a^2} (1 + \frac{\alpha}{2}) [(1 + p^2)^2 + (2 + \frac{p^2}{2})] \\ & + \frac{D_0^2}{a^4} (1 + p^2)(4 + p^2) [\frac{\pi^4}{4} (1 + p^2)(4 + p^2)(1 + \frac{\alpha}{2})^2 + \frac{128\alpha^2}{9}] \} = 0, \end{aligned} \quad (5.189)$$

where $p = a/b$. This is a quadratic in N_x , and hence will have two roots. Of these, the minimum positive root gives the critical load. For a uniform square plate, we have $\alpha = 0$ and $p = 1$. Therefore, we get, on substituting these values in Eq. (5.189),

$$(N_x)_{cr} = 4D_0\pi^2/a^2. \quad (5.190)$$

This result is the same as that obtained in Example 5.1.

For a square plate ($p = 1$) with $\alpha = 1$, we have

$$(N_x)_{cr} = (6.5095)\frac{D_0\pi^2}{a^2}. \quad (5.191)$$

5.5 BUCKLING OF STIFFENED PLATES

5.5.1 Introduction

In a civil engineering structure, whenever large areas have to be roofed without intermediate columns, a grid form of construction is adopted. In this, a slab is laid over beams intersecting each other at right angles. In certain situations, the beams may rest across each other at some other angle. In aircraft construction, the top and bottom surfaces of the wing and the fuselage are made of thin sheets called skins. The skin is generally stiffened by means of stiffeners to prevent its buckling at very low loads. The stiffening can be along one direction or both the directions.

In Sections 5.1–5.4, we saw that the critical value of the normal or shear force is proportional to the flexural rigidity of the plate, and hence the higher the flexural rigidity, the larger the critical load. For a rectangular plate of specified aspect ratio, the critical stress is proportional to t^2/b^2 . The stability of such a plate can be improved either by increasing t or by decreasing b . If the former alternative is adopted, the weight increases, and this is not desirable for aero and space structures. An economical solution is to introduce the stiffeners in the longitudinal direction, thereby decreasing b . The stiffeners so added also increase the weight. But such a weight increase is generally much smaller than that introduced by the increase in the thickness of the plate. The stiffeners can also be placed transversely, but in this case the effect on the buckling strength of a plate is negligible unless these are so positioned that they are very close to each other. In Example 5.1, we noted that a simply-supported uniaxially compressed rectangular plate buckles with a number of half waves, the half waves corresponding to the width b of the plate. Thus, to increase the critical stress of a plate to any considerable extent, the distance between the transverse stiffeners should be as small as possible, preferably far smaller than the plate width.

As can be seen, the presence of a stiffener—longitudinal or transverse—results in abrupt changes in the flexural rigidity of a plate. If the number of stiffeners is large, that is, the distance between the stiffeners is very small as compared to the width of a plate, the stiffened plate then has two different flexural rigidities in the two perpendicular directions (Timoshenko and Krieger [2]).

A structure stiffened in two orthogonal directions in a manner such that the distance between the stiffeners is small is said to be structurally orthotropic. For such a structure, the method of analysis becomes simple and gives a good approximate result. A structure

made up of reinforced concrete, wood, or composite material is called materially orthotropic. This type of structure can be analyzed in the same way as the one that is structurally orthotropic.

In this section, we shall consider numerical examples of stiffened plates, assuming that the deflections are small and the plate material is elastic and isotropic.

EXAMPLE 5.13 (Buckling of simply-supported rectangular plate with central longitudinal stiffener) Consider a rectangular plate of length a and width b and of uniform thickness t . Assume that it is stiffened by a single one-dimensional stiffener of cross-sectional area A_s and second moment of area I_s . Further, suppose that the stiffener is placed on either side of the plate (see Fig. 5.15).

Since the structure is symmetric, the displacement along the Y -axis can be (i) symmetric (when the stiffener deflects) or (ii) antisymmetric (when the stiffener remains straight). In (ii), the nodal line coincides with the axis of the stiffener. This means that the stiffener does not play any part, that is, the plate behaves as a plate of length a and width $b/2$. In this case, the situation is identical to the one we have already discussed (see Example 5.1). We shall therefore analyze the plate for situation (i), i.e., for symmetric mode, where the stringer also contributes to the total deformation.

Since the structure is symmetric about the X -axis, we shall confine our analysis to one half of the plate (it should be noted that the symmetry criterion can be used only when both the boundary conditions and the structure are symmetric). The differential equation governing the response of the plate is (5.38). In terms of the nondimensional parameters ξ and η , defined as

$$x = a\xi, \quad y = \frac{b}{2}\eta, \quad (5.192)$$

Eq. (5.38) can be written as

$$\frac{\partial^4 w}{\partial \xi^4} + \frac{8a^2}{b^2} \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{16a^4}{b^4} \frac{\partial^4 w}{\partial \eta^4} + \frac{N_x a^2}{D} \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.193)$$

Defining $q (=2a/b)$ as the aspect ratio of the plate and

$$\lambda^2 = N_x \frac{a^2}{D} = \sigma_x t \frac{a^2}{Et^3} 12(1 - \nu^2),$$

we can write Eq. (5.193) as

$$\frac{\partial^4 w}{\partial \xi^4} + 2q^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + q^4 \frac{\partial^4 w}{\partial \eta^4} + \lambda^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.194)$$

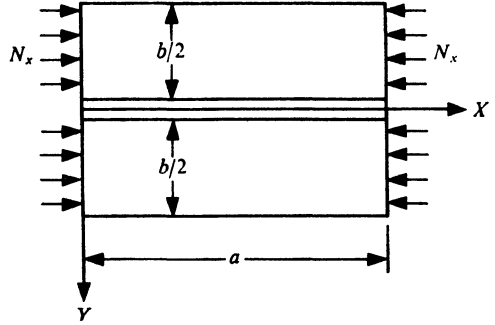


Fig. 5.15 Rectangular Plate with Central Stiffener.

This equation will have the same form as Eq. (5.42) if q is replaced by p . Following the same steps as in Example 5.2, we can write the general solution of Eq. (5.194) as

$$w(\xi, \eta) = \sin m\pi\xi(A_1 \cosh \alpha\eta + A_2 \sinh \alpha\eta + A_3 \cos \beta\eta + A_4 \sin \beta\eta), \quad (5.195)$$

where

$$\begin{aligned} \alpha &= \left[\frac{m^2\pi^2}{q^2} + \left(\lambda^2 \frac{m^2\pi^2}{q^4} \right)^{1/2} \right]^{1/2}, \\ \beta &= \left[-\frac{m^2\pi^2}{q^2} + \left(\lambda^2 \frac{m^2\pi^2}{q^4} \right)^{1/2} \right]^{1/2}. \end{aligned} \quad (5.196)$$

The four constants $A_1, \dots, A_4$ are determined from the boundary conditions

$$w = 0 \quad (\eta = 1), \quad (5.197a)$$

$$\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 1), \quad (5.197b)$$

$$\frac{\partial w}{\partial \eta} = 0 \quad (\eta = 0), \quad (5.197c)$$

$$Q_1 - Q_2 = q_1 \quad (\eta = 0). \quad (5.197d)$$

Here, Q_1 and Q_2 are the shearing forces per unit length of the plate on either side of the stiffener, and can be obtained from the plate transverse shearing forces. Since the stiffener carries the axial load, the differential equation for its deflection w_1 is

$$E_s I_s \frac{\partial^4 w_1}{\partial x^4} + \sigma_c A_s \frac{\partial^2 w_1}{\partial x^2} = q_1, \quad (5.198)$$

where E_s is the elastic modulus of the stiffener material. In terms of the nondimensional parameter ξ , Eq. (5.198) can be written as

$$E_s I_s \frac{\partial^4 w_1}{\partial \xi^4} + \sigma_c A_s a^2 \frac{\partial^2 w_1}{\partial \xi^2} = q_1 a^4. \quad (5.199)$$

The expression for $(Q_1 - Q_2)$ now is

$$a^2(Q_1 - Q_2) = q_1 a^2 = -D \frac{2}{b} \frac{\partial}{\partial \eta} \left[q^2 \frac{\partial^2 w_R}{\partial \eta^2} + (2 - \nu) \frac{\partial^2 w_R}{\partial \xi^2} - q^2 \frac{\partial^2 w_L}{\partial \eta^2} - (2 - \nu) \frac{\partial^2 w_L}{\partial \xi^2} \right], \quad (5.200)$$

where w_R and w_L are the transverse deflection of the plate along the right and the left side of the stringer. In view of the structure symmetry, the following relations exist:

$$\frac{\partial^2 w_R}{\partial \xi^2} = \frac{\partial^2 w_L}{\partial \xi^2} \Big|_{\eta=0}, \quad \frac{\partial^3 w_R}{\partial \eta^3} = - \frac{\partial^3 w_L}{\partial \eta^3} \Big|_{\eta=0}.$$

Substituting these in Eq. (5.200), we have

$$q_1 a^2 = -D \frac{4}{b} \frac{\partial^3 w_R}{\partial \eta^3} \Big|_{\eta=0}. \quad (5.201)$$

For the continuity of the structure, the deflection w_l of the stiffener and the deflection ($w = w_R$) of the plate and their derivatives must be the same at $\eta = 0$. Using Eqs. (5.199) and (5.201) and the condition of continuity of displacement, we can express the boundary condition (5.197d) in terms of w as

$$\left(4\frac{a^2}{b}\frac{\partial^3 w}{\partial \eta^3} + \frac{E_s I_s}{D}\frac{\partial^4 w}{\partial \xi^4} + \frac{\sigma_c A_s}{D}a^2\frac{\partial^2 w}{\partial \xi^2}\right)\bigg|_{\eta=0} = 0. \quad (5.202)$$

Making use of Eq. (5.195) along with the boundary conditions (5.197a), (5.197b), (5.197c), and (5.202), the four simultaneous homogeneous algebraic equations for $A_1, \dots, A_4$ we obtain are

$$A_1 \cosh \alpha + A_2 \sinh \alpha + A_3 \cos \beta + A_4 \sin \beta = 0, \quad (5.203a)$$

$$\alpha^2(A_1 \cosh \alpha + A_2 \sinh \alpha) - \beta^2(A_3 \cos \beta + A_4 \sin \beta) = 0, \quad (5.203b)$$

$$A_2 \alpha + A_4 \beta = 0, \quad (5.203c)$$

$$4\frac{a^2}{b}(A_2 \alpha^3 - A_4 \beta^3) + \left(\frac{E_s I_s}{D}m^4\pi^4 - \frac{\sigma_c A_s}{D}a^2m^2\pi^2\right)(A_1 + A_3) = 0. \quad (5.203d)$$

Defining

$$\delta = \frac{E_s I_s}{Db}, \quad \bar{\gamma} = \frac{A_s}{bI},$$

we can write Eq. (5.203d) as

$$q^2(A_2 \alpha^3 - A_4 \beta^3) + (\delta m^4 \pi^4 - \frac{\sigma_c \bar{\gamma} I}{D}a^2 m^2 \pi^2)(A_1 + A_3) = 0. \quad (5.204)$$

For a nontrivial solution, the determinant of the coefficients of $A_1, \dots, A_4$ must be zero. This leads to the characteristic equation

$$\left(\frac{1}{\alpha} \tanh \alpha - \frac{1}{\beta} \tan \beta\right)(\delta m^4 \pi^4 - \frac{\sigma_c \bar{\gamma} I}{D}a^2 m^2 \pi^2) - 2p(\alpha^2 + \beta^2) = 0, \quad (5.205)$$

where $q = 2p$. Here, the coefficient δ is the ratio of the flexural rigidity of the stiffener to that of the plate of width b , and $\bar{\gamma}$ is the ratio of the cross-sectional area of the stiffener to the area of the plate. The critical load $(N_x)_{cr}$ can now be obtained from the transcendental equation (5.205).

As can be seen from Example 5.13, the differential equation approach is too involved even for a single, symmetrically placed stiffener. The complexity increases manifold as the number of stringers increases. In such a situation, it is advisable that the energy technique be adopted.

EXAMPLE 5.14 (Buckling of simply-supported rectangular plate with longitudinal stiffener) The situation we shall now consider is similar to the one in Example 5.13. However, we shall use the Rayleigh-Ritz technique instead to obtain the solution.

Consider a rectangular plate with the stiffener located at a distance η_l from the edge $\eta = 0$ (see Fig. 5.16a). Clearly, the plate is not symmetric, and therefore we choose the

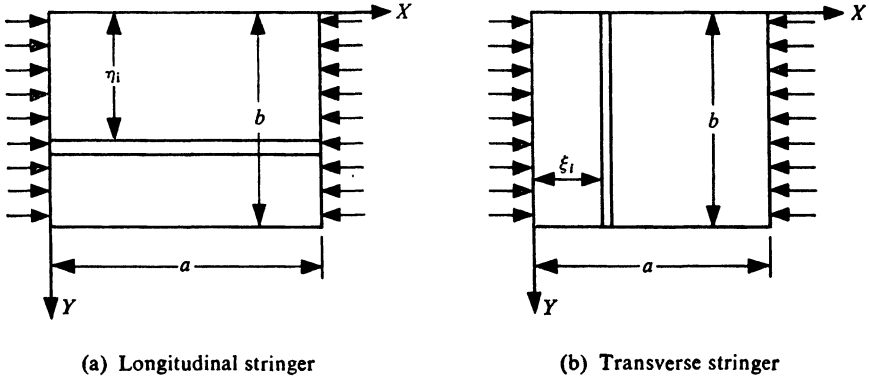


Fig. 5.16 Rectangular Plate with Longitudinal and Transverse Stringers.

coordinate system from the edge of the plate. Further, the strain energy results from the flexure of the plate and the stringer (or stringers). Similarly, the potential energy results from the shortening of the plate and the stringer (or stringers) due to the applied forces. Thus, the change of the plate from an unstiffened to a stiffened state represents the additional contribution of the stringers to the strain and potential energies.

A double Fourier sine series of the following form satisfies the boundary conditions exactly:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi\xi \sin n\pi\eta. \quad (5.206)$$

The strain energy of bending of the plate U_{bp} , from Eq. (5.131), is

$$U_{bp} = \frac{D\pi^4}{8pa^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 (m^2 + p^2 n^2)^2.$$

Denoting $(E_s I_s)_i$ as the flexural rigidity of the i -th stiffener located at a distance η_i from the edge $\eta = 0$, we find the strain energy of bending of the i -th stiffener is

$$\begin{aligned} U_{bai} &= \frac{(E_s I_s)_i}{2a^3} \int_0^1 \left(\frac{\partial^2 w}{\partial \xi^2} \right)_{\eta=\eta_i}^2 d\xi \\ &= \pi^4 \frac{(E_s I_s)_i}{4a^3} \sum_{m=1}^{\infty} m^4 (A_{m1} \sin \pi\eta_i + A_{m2} \sin 2\pi\eta_i + \dots)^2. \end{aligned} \quad (5.207)$$

The potential energy during buckling by the force N_x acting on the plate is given by Eq. (5.132) after putting $N_y = 0$; thus,

$$V_{ep} = -\frac{\pi^2}{8p} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_x A_{mn}^2 m^2. \quad (5.208)$$

The potential energy during buckling by the compressive force P_u acting on the i -th

stringer is

$$V_{esi} = -\frac{P_{si}}{2a} \int_0^1 \left(\frac{\partial w}{\partial \xi} \right)_{\eta=\eta_i}^2 d\xi. \quad (5.209)$$

Substituting for $\partial w / \partial \xi$ and integrating, we get

$$V_{esi} = -\frac{P_{si}\pi^2}{4a} \sum_{m=1}^{\infty} m^2 (A_{m1} \sin \pi\eta_i + A_{m2} \sin 2\pi\eta_i + \dots)^2. \quad (5.210)$$

Defining

$$\delta_i = \frac{(E_s I_s)_i}{D b}, \quad \bar{\gamma}_i = \frac{P_{si}}{b N_x} = \frac{A_{si}}{b h}$$

and summing over all the stringers, we can write the buckling stress σ_{cr} , from the total potential Π_p , as

$$\sigma_{cr} = \frac{\pi^2 D}{b^2 t p^2} \times \frac{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn}^2 (m^2 + p^2 n^2)^2 + 2 \sum_{i=1}^r \delta_i \sum_{m=1}^{\infty} m^4 \left(\sum_{n=1}^{\infty} A_{mn} \sin n\pi\eta_i \right)^2}{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^2 A_{mn}^2 + 2 \sum_{i=1}^r \bar{\gamma}_i \sum_{m=1}^{\infty} m^2 \left(\sum_{n=1}^{\infty} A_{mn} \sin n\pi\eta_i \right)^2}, \quad (5.211)$$

where r indicates the number of stringers. The minimum value of σ_{cr} is determined by taking the derivatives of expression (5.211) and equating them to zero. The set of homogeneous algebraic equations that thus results is

$$\begin{aligned} \frac{\pi^2 D}{b^2 t p^2} [A_{mn} (m^2 + p^2 n^2)^2 + 2 \sum_{i=1}^r \delta_i \sin n\pi\eta_i m^4 \sum_{k=1}^{\infty} a_{mk} \sin k\pi\eta_i \\ - p^2 \sigma_{cr} (m^2 A_{mn} + 2 \sum_{i=1}^r \bar{\gamma}_i \sin n\pi\eta_i m^2 \sum_{k=1}^{\infty} a_{mk} \sin k\pi\eta_i)] = 0. \end{aligned} \quad (5.212)$$

Now, suppose there is only one longitudinal stiffener located at $\eta_1 = 1/2$. Also, as we already know, for the minimum buckling load, a plate buckles with one half sine wave along the X -direction. We shall therefore take $m = 1$, and write the equations for different values of n , the number of half waves along the Y -direction. Thus,

$$\frac{\pi^2 D}{b^2 t p^2} [A_{11} (1 + p^2)^2 + 2\delta (A_{11} - A_{13} + A_{15} - \dots)] - \sigma_{cr} [A_{11} + 2\bar{\gamma} (A_{11} - A_{13} + A_{15})] = 0, \quad (5.213a)$$

$$\frac{\pi^2 D}{b^2 t p^2} A_{12} (1 + 4p^2)^2 - \sigma_{cr} A_{12} = 0, \quad (5.213b)$$

$$\frac{\pi^2 D}{b^2 t p^2} [A_{13} (1 + 9p^2)^2 - 2\delta (A_{11} - A_{13} + A_{15} - \dots)] - \sigma_{cr} [A_{13} - 2\bar{\gamma} (A_{11} - A_{13} + A_{15} - \dots)] = 0, \quad (5.213c)$$

$$\frac{\pi^2 D}{b^2 t p^2} A_{14} (1 + 16p^2)^2 - \sigma_{cr} A_{14} = 0. \quad (5.213d)$$

We can similarly write the equations for other values of n . As can be observed, for even values of n ($= 2, 4, 6, \dots$), the stringer does not contribute to the buckling load since, for such values, the location of the stringer corresponds to the nodal line.

The first approximation to the critical stress can be obtained by taking $n = 1$. This gives

$$\frac{\pi^2 D}{b^2 t p^2} [(1 + p^2)^2 + 2\delta] = \sigma_{cr} (1 + 2\bar{\gamma}). \quad (5.214)$$

Simplifying, we get

$$(N_x)_{cr} = k \frac{\pi^2 D}{b^2}, \quad (5.215)$$

where

$$k = \frac{(1 + p^2)^2 + 2\delta}{p^2(1 + 2\bar{\gamma})}.$$

The next approximation can be had by equating to zero the determinant of the coefficients of A_{11} and A_{13} in Eqs. (5.213a) and (5.213c). The characteristic equation is then given by

$$k^2 p^4 (1 + 4\bar{\gamma}) - k p^2 [(1 + 2\bar{\gamma})(c + d) - 8\bar{\gamma}\delta] + cd - 4\delta^2 = 0, \quad (5.216)$$

where

$$c = (1 + p^2)^2 + 2\delta, \quad d = (1 + 9p^2)^2 + 2\delta.$$

We can now obtain k , and hence $(N_x)_{cr}$, from Eq. (5.216) for definite values of δ , $\bar{\gamma}$, and aspect ratio.

Equation (5.216) is quite general and is applicable for any number of stringers.

EXAMPLE 5.15 (Buckling of transversely stiffened rectangular plate) Consider a rectangular plate of length a and width b , with a transverse stiffener at a distance ξ_l from the X -axis. Assume that the plate is uniaxially compressed and simply-supported on all sides, as shown in Fig. 5.16b. Here again, the plate is not symmetric. Further, the strain energy results from the flexure of the plate and the stringer (or stringers). However, the potential energy results only from the shortening of the plate since the stringer is placed along the Y -axis and the load is acting at the ends of the plate along the X -axis.

Following Example 5.1, we can take the deflection of the plate in the form of a series as

$$w = \sum_{m=1}^{\infty} A_m \sin m\pi\xi \sin \pi\eta \quad (5.217)$$

since, for the maximum buckling load, the plate deforms with one half sine wave along the Y -axis.

The strain energy of bending of the plate U_{bp} , obtained by putting $n = 1$ in Eq. (5.131), is

$$U_{bp} = \frac{D\pi^4}{8pa^2} \sum_{m=1}^{\infty} A_m^2 (m^2 + p^2)^2. \quad (5.218)$$

Denoting $(E_s I_s)_i$ as the flexural rigidity of the i -th stiffener located at a distance ξ_i from the edge $\xi = 0$, we can write the strain energy of bending of the i -th stiffener as

$$\begin{aligned} U_{bi} &= \frac{(E_s I_s)_i}{2b^3} \int_0^1 \left(\frac{\partial^2 w}{\partial \eta^2} \right)_{\xi=\xi_i}^2 d\eta \\ &= \pi^4 \frac{(E_s I_s)_i}{4b^3} \sum_{m=1}^{\infty} A_m^2 \sin^2 m\pi\xi_i. \end{aligned} \quad (5.219)$$

The potential energy of the plate is given as

$$V_{ep} = -\frac{\pi^2}{8p} \sum_{m=1}^{\infty} N_x A_m^2 m^2. \quad (5.220)$$

Defining $\delta_i = (E_s I_s)_i / (Db)$ and summing over all the stringers, we can write the buckling stress σ_{cr} , from the total potential Π_p , as

$$\sigma_{cr} = \frac{D\pi^2}{p^2 b^2 l} \left\{ \frac{A_m^2 (m^2 + p^2)^2 + 2p^3 \sum_{i=1}^r \delta_i \sum A_m^2 \sin^2 m\pi\xi_i}{\sum A_m^2 m^2} \right\}, \quad (5.221)$$

where r indicates the number of transverse stringers. The minimum value of σ_{cr} is determined by taking the derivatives of expression (5.221) and equating them to zero. The resulting set of homogeneous algebraic equations is

$$\begin{aligned} \frac{\pi^2 D}{b^2 l} [A_m (m^2 + p^2)^2 + \sum_{i=1}^r 2\delta_i p^3 \sin m\pi\xi_i (A_1 \sin \pi\xi_i + A_2 \sin 2\pi\xi_i + \dots)] \\ - \sigma_{cr} p^2 m^2 A_m = 0. \end{aligned} \quad (5.222)$$

Now assume there is only one transverse stiffener located at $\xi_1 = 1/2$. We can then write the equations for different values of m , the number of half waves along the X -direction. Thus, for $m = 1$,

$$\frac{\pi^2 D}{b^2 l} [A_1 (1 + p^2)^2 + 2\delta_1 p^3 A_1] = \sigma_{cr} p^2 A_1$$

or

$$(N_x)_{cr} = \frac{\pi^2 D}{b^2 p} [(1 + p^2)^2 + 2\delta_1 p^3], \quad (5.223)$$

and, for $m = 2$,

$$\frac{\pi^2 D}{b^2 l} (4 + p^2)^2 A_2 = \sigma_{cr} p^2 4 A_2$$

or

$$(N_x)_{cr} = \frac{\pi^2 D}{b^2} \frac{(4 + p^2)^2}{4p^2}. \quad (5.224)$$

Equation (5.222) is quite general and is applicable for any number of transverse stiffeners.

EXAMPLE 5.16 (Buckling of simply-supported rectangular stiffened plate) The analysis in Examples 5.14 and 5.15 can be extended to a plate having both longitudinal and transverse stiffeners. If the number of stiffeners is not large, we can then use the Rayleigh-Ritz technique. If there is a large number of equal and equidistant stiffeners, we can consider the stiffened plate as a plate having two different flexural rigidities in the two perpendicular directions. The moment-curvature relations for such an orthotropic plate are given by

$$M_x = -\frac{EI_x}{1 - \nu_x \nu_y} \left(\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2} \right), \quad (5.225a)$$

$$M_y = -\frac{EI_y}{1 - \nu_x \nu_y} \left(\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} \right), \quad (5.225b)$$

$$M_{xy} = 2GI_{xy} \frac{\partial^2 w}{\partial x \partial y}. \quad (5.225c)$$

Substituting for M_x , M_y , and M_{xy} from Eqs. (5.225) in Eq. (5.19) and simplifying, we get

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}, \quad (5.226)$$

where

$$D_x = EI_x / (1 - \nu_x \nu_y),$$

$$D_y = EI_y / (1 - \nu_x \nu_y),$$

$$H = D_x + 2D_{xy}.$$

Assume now that the orthotropic plate is subjected to a uniform compression along the X -axis. Equation (5.226) can then be written as

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} = 0. \quad (5.227)$$

Introducing the nondimensional parameters ξ and η , defined as

$$\xi = x/a, \quad \eta = y/b, \quad (5.228)$$

we find Eq. (5.227) reduces to

$$D_x \frac{\partial^4 w}{\partial \xi^4} + 2Hp^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + D_y p^4 \frac{\partial^4 w}{\partial \eta^4} + N_x a^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.229)$$

Assuming that the plate buckles with one half wave each along the X - and Y -axis, we can write w as

$$w = A \sin \pi \xi \sin \pi \eta. \quad (5.230)$$

Substituting Eq. (5.230) in Eq. (5.229), we obtain

$$D_x \pi^4 + 2H p^2 \pi^4 + D_y p^4 \pi^4 = N_x a^2 \pi^2 \quad (5.231)$$

or

$$(N_x)_{cr} = \frac{\pi^2}{b^2} \left(\frac{D_x}{p^2} + 2H + D_y p^2 \right). \quad (5.232)$$

The smallest value of the critical load is obtained by taking the derivative of Eq. (5.232) with respect to p and equating it to zero. Hence,

$$\frac{(N_x)_{cr}}{p} = 0 = -\frac{2D_x}{p^3} + 2D_y p \quad (5.233)$$

which yields

$$p = (D_x/D_y)^{1/4}. \quad (5.234)$$

The critical load at this value of p is

$$(N_x)_{cr} = \frac{2\pi^2}{b^2} (\sqrt{D_x D_y} + H). \quad (5.235)$$

From Eq. (5.234), we can observe that the minimum buckling load of a simply-supported rectangular plate occurs at an aspect ratio given by the fourth root of the ratio of the flexural stiffnesses along the X - and Y -direction. For an unstiffened plate (as in Example 5.1), the aspect ratio is unity.

5.6 BUCKLING OF CIRCULAR PLATES

In Sections 5.1–5.5, we assumed that the plates are rectangular, and the methods we discussed are suitable only for such a shape. In practice, however, many structures employ circular plates, unstiffened or stiffened. The well-known examples of this shape are corrugated diaphragms (used in pressure vessels), perforated plates (used as base plates in containers), and foundations. The analysis of circular plates becomes simpler if the polar coordinate, rather than the Cartesian coordinate, system is employed. In our discussion here, we shall consider the buckling of circular and annular plates, assuming the material to be linearly elastic and isotropic. For more details on circular plates, see Pandalai and Patel [9] and Vijaykumar and Rao [10].

5.6.1 Governing Equation

Consider a circular plate of uniform cross-section and subjected to a uniform compressive load N_r distributed around the edge of the plate, as shown in Fig. 5.17. We shall limit our discussion to symmetric modes about the axis perpendicular to the plate and passing through its centre.

Let w denote the deflection of the plate in the Z -direction. For small deflection theory, the curvature of the plate in the diametral section is

$$\frac{1}{r_1} = -\frac{d^2 w}{dr^2} = \frac{d\phi}{dr} \quad (5.236)$$

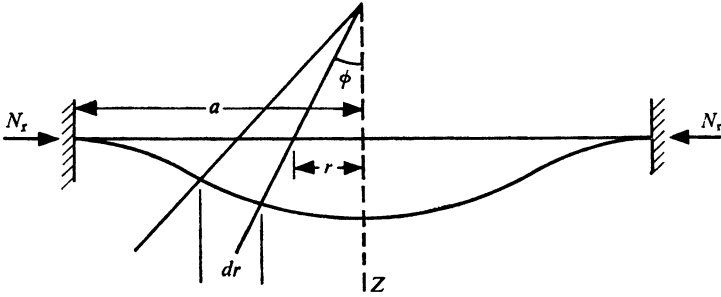


Fig. 5.17 Circular Plate under Compressive Edge Load.

and in the perpendicular direction is

$$\frac{1}{r_2} = \frac{\phi}{r}. \quad (5.237)$$

Substituting for the curvature terms in Eqs. (5.25a) and (5.25b), we get

$$M_r = D\left(\frac{d\phi}{dr} + \nu\frac{\phi}{r}\right), \quad (5.238a)$$

$$M_\theta = D\left(\frac{\phi}{r} + \nu\frac{d\phi}{dr}\right), \quad (5.238b)$$

where M_r and M_θ are the bending moments per unit length acting along the cylindrical and diametral sections, respectively. Considering the equilibrium of an element of the plate shown in Fig. 5.18, we get

$$(M_r + \frac{dM_r}{dr} dr)(r + dr) d\theta - M_r r d\theta - M_\theta dr d\theta + Qr dr d\theta = 0. \quad (5.239)$$

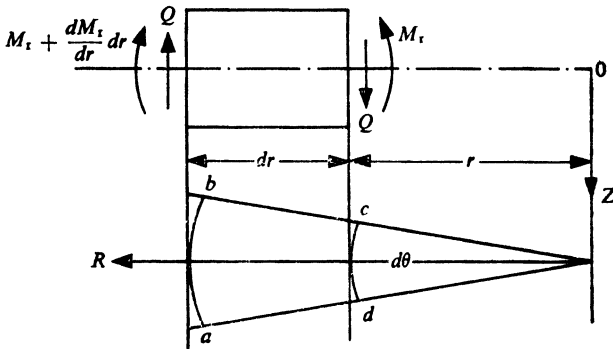


Fig. 5.18 Section of Circular Plate along with Forces.

Neglecting the small quantities of higher order, we obtain

$$M_r + \frac{dM_r}{dr}r - M_\theta + Qr = 0. \quad (5.240)$$

Substituting for M_r and M_θ from Eqs. (5.238), we find Eq. (5.240) reduces to

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{\phi}{r^2} = -\frac{Q}{D}. \quad (5.241)$$

For a uniform compressive load distributed around the plate edge,

$$Q = N_r\phi.$$

Using the notation,

$$\lambda^2 = N_r/D, \quad (5.242)$$

we obtain the governing equation for symmetric buckling of a circular plate as

$$r^2 \frac{d^2\phi}{dr^2} + r \frac{d\phi}{dr} + (\lambda^2 r^2 - 1)\phi = 0. \quad (5.243)$$

Introducing a new variable $u = \lambda r$, we see that Eq. (5.243) becomes

$$u^2 \frac{d^2\phi}{du^2} + u \frac{d\phi}{du} + (u^2 - 1)\phi = 0. \quad (5.244)$$

Equation (5.244) is the well-known Bessel equation, the most general solution of which can be written as

$$\phi = AJ_1(u) + BY_1(u), \quad (5.245)$$

where $J_1(u)$ and $Y_1(u)$ are the Bessel functions of first order of the first and second kinds, respectively. The coefficients A and B are obtained from the boundary conditions.

We shall now demonstrate the application of Eq. (5.245) to clamped and simply-supported plates.

Clamped plate

The boundary conditions are as follows:

(i) At the centre of the plate ($r = u = 0$), the angle ϕ must be zero to satisfy the condition of symmetry. This makes $B = 0$ since as $r \rightarrow 0$, $Y_1(r) \rightarrow \infty$.

(ii) The slope at the edge of the plate is zero, i.e., $\phi|_{r=a} = 0$.

For a nontrivial solution of Eq. (5.245), $J_1(\lambda a) = 0$. From the table of functions, the smallest root of this equation is

$$\lambda a = 3.882. \quad (5.246)$$

Substituting for λ in Eq. (5.242), we obtain

$$(N_r)_{cr} = 14.68D/a^2. \quad (5.247)$$

Simply-supported plate

The boundary conditions are as follows:

(i) From the symmetry of deflection,

$$B = 0. \quad (5.248a)$$

(ii)

$$M_r = \left(\frac{d\phi}{dr} + \nu \frac{\phi}{r} \right)_{r=a} = 0. \quad (5.248b)$$

Substituting for ϕ , we see that relation (5.248b) reduces to

$$\lambda A_1 \left[\frac{d}{du} J_1(u) + \nu \frac{J_1(u)}{u} \right]_{r=a} = 0. \quad (5.249)$$

Using the derivative formula

$$\frac{dJ_1}{du} = J_0 - \frac{J_1}{u}, \quad (5.250)$$

the characteristic equation we get is

$$\lambda a J_0(\lambda a) - (1 - \nu) J_1(\lambda a) = 0. \quad (5.251)$$

The smallest root of this transcendental equation for $\nu = 0.3$ is 2.05. Hence,

$$(N_r)_{cr} = \frac{4.20D}{a^2}. \quad (5.252)$$

5.6.2 Buckling of Orthotropic Circular Plates

A circular plate fabricated with radial and/or circumferential reinforcements may be mathematically idealized as an orthotropic plate of uniform thickness t having different material properties in the radial and circumferential directions.

The stress-strain relations for an orthotropic circular plate are

$$\sigma_r = E_r \epsilon_r + E_{r\theta} \epsilon_\theta, \quad (5.253a)$$

$$\sigma_\theta = E_r \epsilon_r + E_\theta \epsilon_\theta, \quad (5.253b)$$

$$\tau_{r\theta} = G_{r\theta} \gamma_{r\theta}, \quad (5.253c)$$

where E_r , E_θ are the elastic moduli along the radial, circumferential directions, and $E_{r\theta}$ is the coupled modulus.

Equations (5.25) give the moment-curvature relations for a plate in the Cartesian coordinate system. The moment-curvature relations for a plate in the polar coordinate system are

$$M_r = -D_r \left[\frac{\partial^2 w}{\partial r^2} + \alpha \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right], \quad (5.254a)$$

$$M_\theta = -D_r[\beta(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}) + \alpha\frac{\partial^2 w}{\partial \theta^2}], \quad (5.254b)$$

$$M_{r\theta} = -2\delta D_r \frac{d}{dr}(\frac{1}{r}\frac{\partial w}{\partial \theta}), \quad (5.254c)$$

where

$$D_r = \frac{E_r t^3}{12}, \quad \alpha = \frac{E_r \epsilon_\theta}{E_r}, \quad \beta = \frac{E_\theta}{E_r}, \quad \delta = \frac{g_{r\theta}}{E_r}.$$

The governing equation for a circular plate subjected to a lateral load $q(r, \theta)$ is obtained by rewriting Eq. (5.19) in terms of the polar coordinates and replacing the last three terms in the left-hand side of Eq. (5.19) by the lateral load $q(r, \theta)$. Thus,

$$\frac{d^2}{dr^2}(rM_r) - \frac{d}{dr}M_\theta + \frac{1}{r}[\frac{2d}{dr}(r\frac{d}{d\theta}M_{r\theta}) + \frac{d^2}{d\theta^2}M_\theta] = -rq(r, \theta). \quad (5.255)$$

Substituting for M_r , M_θ , and $M_{r\theta}$ from Eqs. (5.254) in Eq. (5.255), we get the governing equation as

$$\begin{aligned} r\frac{\partial^4 w}{\partial r^4} + 2\frac{\partial^3 w}{\partial r^3} - \beta[\frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}] + \frac{\beta}{r^3}[\frac{\partial^4 w}{\partial \theta^4} + 2\frac{\partial^2 w}{\partial \theta^2}] \\ + 2(\alpha + 2\delta)[\frac{1}{r}\frac{\partial^4 w}{\partial \theta^2 \partial r^2} - \frac{1}{r^2}\frac{\partial^3 w}{\partial \theta^2 \partial r} + \frac{1}{r^3}\frac{\partial^2 w}{\partial \theta^2}] = -\frac{r}{D_r}q(r, \theta). \end{aligned} \quad (5.256)$$

For axisymmetric buckling, $w = w(r)$ and the load term $q(r, \theta)$ can be expressed as

$$q = N_r \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} N_\theta \frac{\partial w}{\partial r}, \quad (5.257)$$

where $N_r = t\sigma_r$ and $N_\theta = t\sigma_\theta$ are the stress resultants induced by the applied uniform edge load. The resultant loads N_r and N_θ are obtained by analyzing an orthotropic circular plate subjected to a uniform compression σ_0 at $r = a$.

The equilibrium equation for a plate, in the absence of body forces, is given as (Timoshenko and Goodier [5])

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \quad (5.258)$$

The compatibility condition is given by

$$\frac{d}{dr}(r\epsilon_\theta) - \epsilon_r = 0. \quad (5.259)$$

Substituting for ϵ_r and ϵ_θ from Eqs. (5.253) and making use of Eq. (5.258), we see that Eq. (5.259) gives the compatibility in terms of the stresses as

$$r^2 \frac{d^2 \sigma_r}{dr^2} + 3r \frac{d\sigma_r}{dr} + (1 - \beta)\sigma_r = 0. \quad (5.260)$$

Substituting $\sigma_r = r^\lambda$ in Eq. (5.260), we get the characteristic equation as

$$\lambda^2 + 2\lambda + 1 - \beta = 0. \quad (5.261)$$

The roots of this equation are

$$\lambda = \pm(\sqrt{\beta} - 1).$$

The most general solution of Eq. (5.260) is therefore

$$\sigma_r = Ar^{\sqrt{\beta}-1} + Br^{-(\sqrt{\beta}-1)}. \quad (5.262)$$

The constants A and B are obtained from the conditions σ_r is finite at $r = 0$ and $\sigma_r(r = a) = \sigma_0$. From these conditions, we get

$$\sigma_r = \sigma_0 \left(\frac{r}{a}\right)^{\sqrt{\beta}-1}. \quad (5.263)$$

Hence,

$$\sigma_\theta = \sqrt{\beta} \sigma_0 \left(\frac{r}{a}\right)^{\sqrt{\beta}-1}. \quad (5.264)$$

Therefore, N_r and N_θ are given by

$$N_r = t \sigma_0 \left(\frac{r}{a}\right)^{\sqrt{\beta}-1}, \quad (5.265a)$$

$$N_\theta = \sqrt{\beta} t \sigma_0 \left(\frac{r}{a}\right)^{(\sqrt{\beta}-1)}. \quad (5.265b)$$

The governing equation (5.256) can be rewritten as

$$\frac{r d^4 w}{dr^4} + 2 \frac{d^3 w}{dr^3} - \left(\frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr}\right) = -\left(\frac{1}{r} \frac{d^2 w}{dr^2} + \frac{\sqrt{\beta}}{r^2} \frac{dw}{dr}\right) k^2 r^{\sqrt{\beta}+1}, \quad (5.266)$$

where $k^2 = t \sigma_0 / (D_r a^{2p})$ and $2p = \sqrt{\beta} - 1$. Integrating Eq. (5.266) once, we get

$$\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} + \frac{1}{r^2} [(k_r^{1+p})^2 - \beta] \frac{dw}{dr} = 0. \quad (5.267)$$

Let

$$w(r) = \sum_{n=0}^{\infty} A_n (k r^{1+p})^n. \quad (5.268)$$

Substituting relation (5.268) in Eq. (5.267), we get the recurrence relation as

$$A_n = -A_{n-2} / \{n(1+p)[n(1+p) + 2p]\}. \quad (5.269)$$

From the condition that the stresses at the centre of the plate are finite, we have $A_1 = 0$.

Hence,

$$w = A_0 + A_2 \sum_{m=2, 4, 6}^{\infty} C_m X^m, \quad (5.270)$$

where $X = kr^{(1+p)}$.

Clamped plate

The boundary conditions are

$$w = \frac{dw}{dr} = 0 \quad (r = a).$$

The characteristic equation is

$$\sum_{m=2, 4, 6}^{\infty} m C_m X^m|_{r=a} = 0. \quad (5.271)$$

Hence,

$$N_{cr} = D_r \frac{\lambda_{cr}^2}{a^2},$$

where λ_{cr} is the lowest root of the characteristic equation (5.271). For $p = 0, 0.05$, and 1 , the roots are $14.66, 16.77$, and 18.88 , respectively.

Simply-supported plate

The boundary conditions are

$$w = M_r = 0 \quad (r = a).$$

The characteristic equation is

$$\sum_{m=2, 4, 6}^{\infty} m[m(1+p) - (1-\alpha)] C_m X^m|_{r=a} = 0.$$

Here, the values of λ_{cr}^2 for $p = 0, 0.05$, and 0.1 are $4.28, 4.94$, and 5.64 , respectively.

5.7 BUCKLING OF THICK RECTANGULAR PLATES

Strictly speaking, a plate has to be analyzed treating the plate as a three-dimensional problem. Because of the mathematical complexities involved, such a problem is reduced either to a two-dimensional or one-dimensional one by making certain simplifying assumptions. The buckling of plates belongs to the class of three-dimensional problems and when discussing it (see Sections 5.1–5.5), we made one such assumption, namely, the thickness of the plate is very small as compared to the other two characteristic dimensions. This assumption requires the satisfaction of only two boundary conditions, though there are three such conditions, at each edge of a plate.

In thin plate theory, the transverse shear deformation is neglected. Therefore, when this theory is applied, the errors in the plate response increase as the plate thickness increases (Reissner [11]). In view of the current extensive use of thick plates, e.g., in ship hulls, end

covers of cylinders, and water tanks, it is advisable we know how such plates are analyzed on the basis of three-dimensional elasticity. In this section, we shall consider the buckling of thick isotropic plates.

5.7.1 Buckling of Thick Isotropic Plates

Consider a thick, homogeneous rectangular plate subjected to an inplane load N_x at $x = 0$, a , as shown in Fig. 5.19. In addition to the small deformation assumptions, suppose that

- (i) the plate material is homogeneous and isotropic and
- (ii) the stress-strain relation is linear.

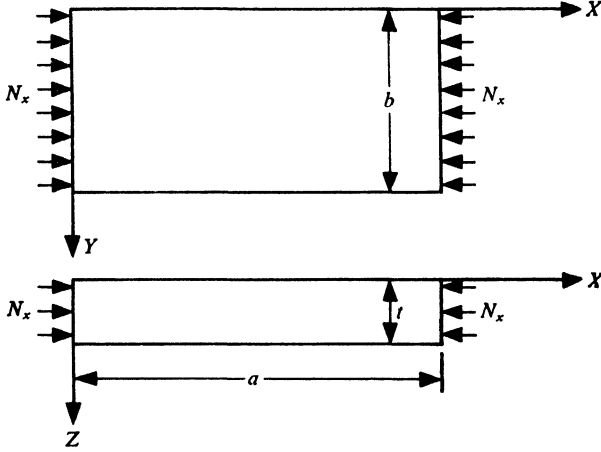


Fig. 5.19 Thick Rectangular Plate under Uniaxial Compressive Loading.

Our method of treatment here follows the technique presented in Reissner [11].

In the three-dimensional Cartesian coordinate system, the equilibrium equations, in the absence of body forces, are (Love [12])

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad (5.272a)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad (5.272b)$$

$$\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \quad (5.272c)$$

Also, the strain-displacement relations are given by

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad (5.272d)$$

$$\epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{yz} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}, \quad (5.272e)$$

$$\epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \quad (5.272f)$$

The stress-strain relations are given by

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)], \quad (5.272g)$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)], \quad (5.272h)$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_y + \sigma_x)], \quad (5.272i)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad (5.272j)$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz}, \quad (5.272k)$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}. \quad (5.272l)$$

Making use of Eqs. (5.272) along with the compatibility conditions, we get the following equilibrium equations in terms of the displacements and the applied force N_x :

$$\frac{G}{1-2\nu} \frac{\partial e}{\partial x} + G\nabla^2 u = N_x \frac{\partial^2 u}{\partial x^2}, \quad (5.273a)$$

$$\frac{G}{1-2\nu} \frac{\partial e}{\partial y} + G\nabla^2 v = N_x \frac{\partial^2 v}{\partial x^2}, \quad (5.273b)$$

$$\frac{G}{1-2\nu} \frac{\partial e}{\partial z} + G\nabla^2 w = N_x \frac{\partial^2 w}{\partial x^2}, \quad (5.273c)$$

where

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The boundary conditions along the edges of a simply-supported plate are

$$\sigma_x = 0, \quad w = 0, \quad \frac{\partial v}{\partial z} = 0 \quad (x = 0, a), \quad (5.274a)$$

$$\sigma_y = 0, \quad w = 0, \quad \frac{\partial u}{\partial z} = 0 \quad (y = 0, b). \quad (5.274b)$$

A characteristic equation for the buckling stress can be obtained by choosing suitable displacement functions for u , v , and w , which satisfy the differential equations (5.273), the edge boundary conditions (5.274), and the lateral surface boundary conditions

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \quad (z = 0, t). \quad (5.275)$$

By choosing u , v , and w as

$$u = t \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \phi_{mn}(\xi), \quad (5.276a)$$

$$v = t \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \psi_{mn}(\xi), \quad (5.276b)$$

$$w = t \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \chi_{mn}(\xi), \quad (5.276c)$$

where $\xi = z/t$, the edge boundary conditions (5.274) are identically satisfied. Substituting u , v , and w in Eqs. (5.273), we get

$$\frac{d^2\phi}{d\xi^2} + \frac{\mu}{1-2\nu} \frac{d\chi}{d\xi} - (r^2 + \frac{\mu^2}{1-2\nu})\phi - \frac{\mu\sigma}{1-2\nu}\psi = 0, \quad (5.277a)$$

$$\frac{d^2\psi}{d\xi^2} + \frac{\sigma}{1-2\nu} \frac{d\chi}{d\xi} - \frac{\mu\sigma}{1-2\nu}\phi - (r^2 + \frac{\sigma^2}{1-2\nu})\psi = 0, \quad (5.277b)$$

$$-\frac{2(1-\nu)}{1-2\nu} \frac{d^2\chi}{d\xi^2} + \frac{\mu}{1-2\nu} \frac{d\phi}{d\xi} + \frac{\sigma}{1-2\nu} \frac{d\psi}{d\xi} + r^2\chi = 0, \quad (5.277c)$$

where

$$\mu = \frac{m\pi t}{a}, \quad \sigma = \frac{n\pi t}{b}, \quad g^2 = \mu^2 + \sigma^2, \quad \rho = \frac{N_x}{G}, \quad r^2 = g^2 - \rho\mu^2$$

and m and n are the number of half waves along the X - and Y -direction.

The differential equations (5.277) can be written in the matrix form as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-2(1-\nu)}{1-2\nu} \end{bmatrix} \{\bar{v}\}'' + \begin{bmatrix} 0 & 0 & \frac{\mu}{1-2\nu} \\ 0 & 0 & \frac{\sigma}{1-2\nu} \\ \frac{\mu}{1-2\nu} & \frac{\sigma}{1-2\nu} & 0 \end{bmatrix} \{\bar{v}\}' + \begin{bmatrix} -(r^2 + \frac{\mu^2}{1-2\nu}) & -\frac{\mu\sigma}{1-2\nu} & 0 \\ -\frac{\mu\sigma}{1-2\nu} & -(r^2 + \frac{\sigma^2}{1-2\nu}) & 0 \\ 0 & 0 & r^2 \end{bmatrix} \{\bar{v}\} = 0, \quad (5.278)$$

where $\{\bar{v}\} = [\phi \ \psi \ \chi]^T$. Substituting $\{\bar{v}\} = \{A\}e^{\lambda\xi}$ in Eq. (5.278), we obtain a set of algebraic equations. Thus, for a nontrivial solution of these equations, the determinant of the coefficients of $\{A\}$ must be zero. This gives

$$\begin{vmatrix} \lambda^2 - (r^2 + \frac{\mu^2}{1-2\nu}) & -\frac{\mu\sigma}{1-2\nu} & \lambda\frac{\mu}{1-2\nu} \\ -\frac{\mu\sigma}{1-2\nu} & \lambda^2 - (r^2 + \frac{\sigma^2}{1-2\nu}) & \lambda\frac{\sigma}{1-2\nu} \\ \frac{\lambda\mu}{1-2\nu} & \lambda\frac{\sigma}{1-2\nu} & \frac{-2(1-\nu)\lambda^2 + r^2}{1-2\nu} \end{vmatrix} = 0. \quad (5.279)$$

Expanding this determinant, we get

$$(\lambda^2 - r^2)^2(\lambda^2 - s^2) = 0, \quad (5.280)$$

where

$$s^2 = g^2 - \rho\mu^2 \frac{1-2\nu}{2(1-\nu)}.$$

Equation (5.280) has in all six roots, namely, two repeated roots given by $+r$ and $-r$ and two simple roots $+s$ and $-s$. The vectors associated with these roots can be obtained from Eq. (5.279) after substituting the roots corresponding to λ . A sample calculation for obtaining the eigenvectors is as follows.

Let the root corresponding to λ be $+r$. Equation (5.279) can then be rewritten as

$$\begin{bmatrix} -\frac{\mu^2}{1-2\nu} & -\frac{\mu\sigma}{1-2\nu} & \frac{\mu r}{1-2\nu} \\ -\frac{\sigma\mu}{1-2\nu} & -\frac{\sigma^2}{1-2\nu} & \frac{r\sigma}{1-2\nu} \\ \frac{\mu r}{1-2\nu} & \frac{r\sigma}{1-2\nu} & -\frac{r^2}{1-2\nu} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = 0. \quad (5.281)$$

After manipulation, Eq. (5.281) can be reduced to

$$\begin{bmatrix} -\frac{\mu^2}{1-2\nu} & -\frac{\mu\sigma}{1-2\nu} & \frac{\mu r}{1-2\nu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = 0. \quad (5.282)$$

This implies

$$(-\mu a_1 - \sigma a_2 + r a_3) = 0. \quad (5.283)$$

Equation (5.283) has two independent solutions, namely,

$$\begin{aligned} a_1 &= 1, & a_2 &= 0, & a_3 &= \mu/r, \\ a_1 &= 1, & a_2 &= 1, & a_3 &= \sigma/r. \end{aligned}$$

Hence, the eigenvectors corresponding to $\lambda = r$ are

$$\{A_1\} = \begin{Bmatrix} 1 \\ 0 \\ \mu/r \end{Bmatrix} \quad \text{or} \quad \{A_1\} = \begin{Bmatrix} r \\ 0 \\ \mu \end{Bmatrix}$$

and

$$A_2 = \begin{Bmatrix} 0 \\ r \\ \sigma \end{Bmatrix}.$$

Similarly, we can obtain the eigenvectors corresponding to $\lambda = -r, s, -s$.

The general solution of Eq. (5.279) can then be written as

$$\phi = \alpha_1 \begin{Bmatrix} r \\ 0 \\ \mu \end{Bmatrix} e^{r\xi} + \alpha_2 \begin{Bmatrix} 0 \\ r \\ \sigma \end{Bmatrix} e^{r\xi} + \alpha_3 \begin{Bmatrix} r \\ 0 \\ -\mu \end{Bmatrix} e^{-r\xi} + \alpha_4 \begin{Bmatrix} 0 \\ r \\ -\sigma \end{Bmatrix} e^{-r\xi} + \alpha_5 \begin{Bmatrix} \mu \\ \sigma \\ s \end{Bmatrix} e^{s\xi} + \alpha_6 \begin{Bmatrix} \mu \\ \sigma \\ -s \end{Bmatrix} e^{-s\xi}. \quad (5.284)$$

In the expanded form, ϕ can be written as

$$\begin{aligned} \phi &= r(\alpha_1 e^{r\xi} + \alpha_3 e^{-r\xi}) + \mu(\alpha_5 e^{s\xi} + \alpha_6 e^{-s\xi}) \\ &= r[(\alpha_1 + \alpha_3) \cosh r\xi + (\alpha_1 - \alpha_3) \sinh r\xi] + [(\alpha_5 + \alpha_6) \cosh s\xi + (\alpha_5 - \alpha_6) \sinh s\xi] \\ &= r(\beta_1 \cosh r\xi + \beta_2 \sinh r\xi) + \mu(\beta_5 \cosh s\xi + \beta_6 \sinh s\xi). \end{aligned} \quad (5.285a)$$

Similarly,

$$\psi = r(\beta_3 \cosh r\xi + \beta_4 \sinh r\xi) + \sigma(\beta_5 \cosh s\xi + \beta_6 \sinh s\xi), \quad (5.285b)$$

$$\begin{aligned} \chi &= u(\beta_1 \sinh r\xi + \beta_2 \cosh r\xi) + \sigma(\beta_3 \sinh r\xi + \beta_4 \cosh r\xi) \\ &\quad + s(\beta_5 \sinh s\xi + \beta_6 \cosh s\xi), \end{aligned} \quad (5.285c)$$

where

$$\begin{aligned} \beta_1 &= \alpha_1 + \alpha_3, & \beta_2 &= \alpha_1 - \alpha_3, & \beta_3 &= \alpha_2 + \alpha_4, & \beta_4 &= \alpha_2 - \alpha_4, \\ \beta_5 &= \alpha_5 + \alpha_6, & \beta_6 &= \alpha_5 - \alpha_6. \end{aligned}$$

The lateral surface boundary conditions (5.275) can be written in terms of the displacements u , v , and w as

$$\sigma_z = 0 \Rightarrow [(1 - \nu) \frac{\partial w}{\partial z} + \nu \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}] = 0, \quad (5.286a)$$

$$\tau_{xz} = 0 \Rightarrow \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = 0, \quad (5.286b)$$

$$\tau_{yz} = 0 \Rightarrow \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = 0. \quad (5.286c)$$

Substituting for u , v , and w from Eqs. (5.285) in Eqs. (5.286), we get a set of six simultaneous homogeneous algebraic equations. For a nontrivial solution of these equations, the determinant of the coefficients must be zero. This leads to the characteristic equation

$$8g^2rs(r^2 + g^2)^2(1 - \cosh r \cosh s) + \{16g^4r^2s^2 + (r^2 + g^2)^4\} \sinh r \sinh s = 0. \quad (5.287)$$

The critical load N_x , contained in r and s , is obtained by solving this equation. The thin plate analysis (disregarding the thickness effect) gives the buckling load $(N_x)_{cr}$ as [see Eq. (5.55)]

$$(N_x)_{cr} = \frac{kD\pi^2}{b^2},$$

where $k = 4.0$. Equation (5.287) can also be used to predict the critical load where the thickness effect is not neglected. However, in such a situation, the value of k is different. The values of k for $t/b = 0.0, 0.05, 0.1, 0.2$ are 4.0, 3.911, 3.741, 3.150, respectively. This implies that a thick plate buckles earlier than a thin plate of the same dimensions and boundary conditions as the thick plate. Therefore, for purposes of design, the effect of thickness on the buckling load must be taken into account.

For more details on thick plates, see Srinivas and Rao [13, 14].

5.8 NUMERICAL TECHNIQUES

In Sections 5.1–5.7, we observed that the analysis of a plate is quite complicated since the plate material can be isotropic, orthotropic, or anisotropic. In addition, the plate may have a number of layers. Further, the plate may have any shape (e.g., circular, rectangular, and skew), complicated boundary conditions (e.g., partial fixity, elastic support, and re-entrant corners), and varying stiffness and edge loading conditions. Such a plate often ends up in a partial differential equation with variable coefficients and rarely lends itself to a closed-form solution. In this situation, the application of even the Rayleigh-Ritz or the Galerkin technique leads to a very approximate solution since each of these methods requires the knowledge of displacement function which itself is approximately chosen. Of course, either of the two techniques can be profitably used if the displacement function selected is such that it contains a large number of terms.

In Chapter 1, we discussed two numerical techniques, namely, the finite difference and finite element techniques, and their application to one-dimensional structures such as beams and columns. The application of these two techniques to a plate is a little more involved since the order of the equation governing the plate behaviour is higher and the boundary conditions may be complicated. In this section, we shall discuss the two techniques in the context of the plate problems.

5.8.1 Finite Difference Technique

The finite difference approximation requires the replacement of a differential equation by a difference equation with finite degrees of freedom. Such an approximation leads to a set

of simultaneous equations involving deflections at the grid points. For obtaining the difference equation, central differences are used (see Section 1.9.1). Since the convergence of the solution of a difference equation is of the order of h^2 (h being the grid spacing), the accuracy of the solution can be improved by using an extrapolation technique and higher order finite differences (Szilord [15]). This however may require finding the fictitious points outside the domain of a structure. To extend the finite difference technique to a plate, it is necessary to evolve expressions for the differences corresponding to the partial derivatives in the governing equation of the plate.

Consider a plate represented by a network of discrete points, as shown in Fig. 5.20. The points are evenly placed at a distance h along the X - and Y -direction. In many situations,

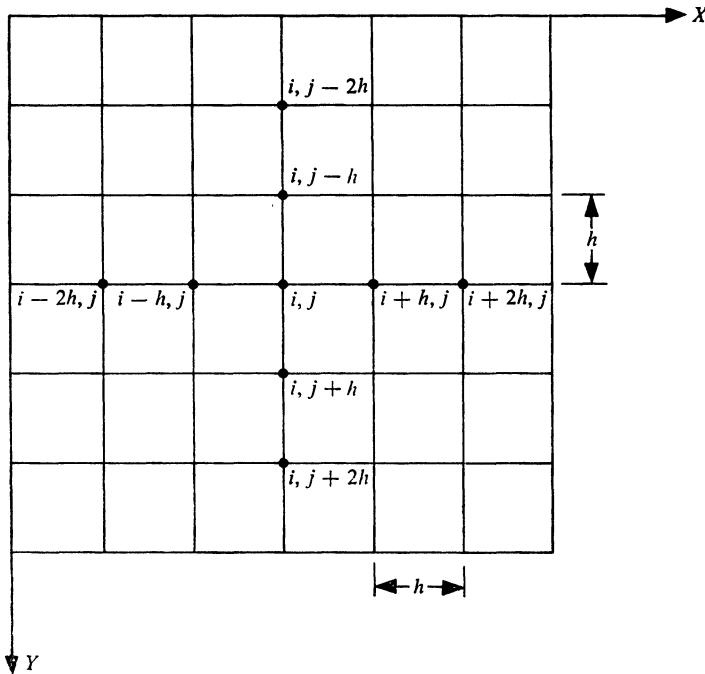


Fig. 5.20 Finite Difference Mesh for Plates.

it may not be necessary to have the same distance along the X - and Y -direction. Following Eq. (1.213), we can express the difference representation for the second and fourth derivatives along the X -direction as

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j} \approx \Delta^2 w_{i,j} = (w_{i+h,j} - 2w_{i,j} + w_{i-h,j})/h^2, \quad (5.288a)$$

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{i,j} \approx \Delta^4 w_{i,j} = (w_{i+2h,j} - 4w_{i+h,j} + 6w_{i,j} - 4w_{i-h,j} + w_{i-2h,j})/h^4. \quad (5.288b)$$

The second and fourth differences with respect to y can be written by keeping x fixed and moving along y . Thus,

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} = (w_{i,j+h} - 2w_{i,j} + w_{i,j-h})/h^2, \quad (5.288c)$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_{i,j} = (w_{i,j+2h} - 4w_{i,j+h} + 6w_{i,j} - 4w_{i,j-h} + w_{i,j-2h})/h^4. \quad (5.288d)$$

The cross derivative with respect to x and y can be written by taking the second difference of Eq. (5.288c) with respect to x . Thus,

$$\begin{aligned} \left(\frac{\partial^2 w}{\partial x^2 \partial y^2}\right)_{i,j} &= \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} \approx \frac{1}{h^2} \left[\left(\frac{\partial^2 w}{\partial y^2}\right)_{i+h,j} - 2\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} + \left(\frac{\partial^2 w}{\partial y^2}\right)_{i-h,j} \right] \\ &\approx \frac{1}{h^4} [(w_{i+h,j+h} - 2w_{i+h,j} + w_{i+h,j-h}) - 2(w_{i,j+h} - 2w_{i,j} + w_{i,j-h}) \\ &\quad + (w_{i-h,j+h} - 2w_{i-h,j} + w_{i-h,j-h})]. \end{aligned} \quad (5.288e)$$

As already stated (see Section 1.9.2), each of these differences has an inherent error of the order of h^2 . In the governing equation of a plate, these differences replace the derivatives corresponding to them.

We shall now illustrate the technique by solving typical plate problems.

EXAMPLE 5.17 (Buckling of square plate under uniform axial loading) Consider a square plate with the loaded edges simply-supported and the remaining two edges clamped, as shown in Fig. 5.21. The governing differential equation for such a plate can be written, following Eq. (5.27), as

$$\nabla^4 w + \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2} = 0. \quad (5.289)$$

The boundary conditions are

$$\begin{aligned} w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad (x = 0, a), \\ w = 0, \quad \frac{\partial w}{\partial y} = 0 \quad (y = 0, a). \end{aligned} \quad (5.290)$$

Introducing the nondimensional parameters ξ and η , defined as

$$\xi = x/a, \quad \eta = y/b,$$

we can write Eq. (5.289) as

$$\nabla^4 w + N_x \frac{a^2}{D} \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (5.291)$$

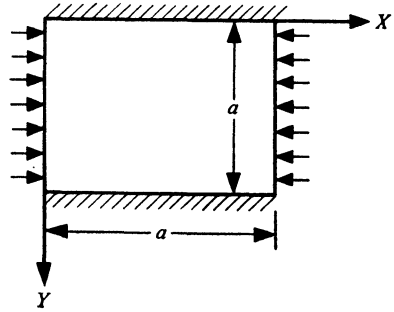


Fig. 5.21 Square Plate under Axial Compressive Loading.

If the plate is covered with a square lattice of mesh size $1/n$, then, by multiplying Eq. (5.291) by $h^4 = 1/n^4$ and writing the differences for the second and fourth derivatives, we have

$$w_{i+2,j} - 8w_{i+1,j} + 20w_{i,j} - 8w_{i-1,j} + w_{i-2,j} + 2w_{i+1,j+1} + 2w_{i+1,j-1} - 8w_{i,j+1} - 8w_{i,j-1} + w_{i,j+2} + w_{i,j-2} + 2w_{i-1,j+1} + 2w_{i-1,j-1} + \lambda_n(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}) = 0, \quad (5.292)$$

where $\lambda_n = N_x a^2 / (Dn^2)$.

Applying Eq. (5.292) at the internal pivotal points $(i, j = 1, 2, \dots, n-1)$, we get a set of homogeneous algebraic equations in the unknown pivotal displacements $w_{i,j}$. For a non-trivial solution, the determinant of the coefficients of the displacements $w_{i,j}$ is identically equal to zero. The lowest value of N_x , which makes the determinant zero, gives the first critical load. Writing the boundary conditions (5.290) in terms of the differences, we have

$$\begin{aligned} w_{i,j} &= 0, & w_{i+1,j} &= -w_{i-1,j} & (i = 0, n, j = 0, 1, 2, \dots, n), \\ w_{i,j} &= 0, & w_{i,j+1} &= w_{i,j-1} & (i = 1, 2, \dots, n, j = 0, n). \end{aligned} \quad (5.293)$$

For $n = 3$, the finite difference mesh for the plate is as shown in Fig. 5.22. Writing the

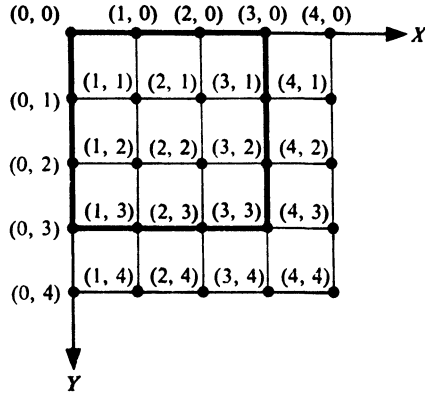


Fig. 5.22 Finite Difference Mesh for Plate ($n = 3$).

difference equations for the interior points $(1, 1)$, $(2, 1)$, $(1, 2)$, and $(2, 2)$, the set of homogeneous algebraic equations we get is

$$\begin{aligned} w_{3,1} - 8w_{2,1} + 20w_{1,1} - 8w_{0,1} + w_{-1,1} + 2w_{2,2} + 2w_{2,0} - 8w_{1,2} - 8w_{1,0} + w_{1,3} + w_{1,-1} \\ + 2w_{0,1} + 2w_{0,0} + \lambda_3(w_{2,1} - 2w_{1,1} + w_{0,1}) &= 0, \\ w_{3,2} - 8w_{2,2} + 20w_{1,2} - 8w_{0,2} + w_{-1,2} + 2w_{2,3} + 2w_{2,1} - 8w_{1,3} - 8w_{1,1} + w_{1,4} + w_{1,0} \\ + 2w_{0,2} + 2w_{0,1} + \lambda_3(w_{2,2} - 2w_{1,2} + w_{0,2}) &= 0, \\ w_{4,1} - 8w_{3,2} + 20w_{2,1} - 8w_{1,1} + w_{0,1} + 2w_{3,2} + 2w_{3,0} - 8w_{2,2} - 8w_{2,0} \\ + w_{2,3} + w_{2,-1} + 2w_{1,1} + 2w_{1,0} + \lambda_3(w_{3,1} - 2w_{2,1} + w_{1,1}) &= 0, \end{aligned}$$

$$w_{4,2} - 8w_{3,2} + 20w_{2,2} - 8w_{1,2} + w_{0,2} + 2w_{3,3} + 2w_{3,1} - 8w_{2,3} - 8w_{2,1} + w_{2,4} \\ + w_{2,0} + 2w_{1,2} + 2w_{1,1} + \lambda_3(w_{3,2} - 2w_{3,2} + w_{1,2}) = 0.$$

Incorporating the boundary values (5.293), we find the foregoing equations reduce to

$$\begin{aligned} (20 - 2\lambda_3)w_{1,1} - 8w_{1,2} + (\lambda_3 - 8)w_{2,1} + 2w_{2,2} &= 0, \\ -8w_{1,1} + (20 - 2\lambda_3)w_{1,2} + 2w_{2,1} + (\lambda_3 - 8)w_{2,2} &= 0, \\ (\lambda_3 - 8)w_{1,1} + 2w_{1,2} + (20 - 2\lambda_3)w_{2,1} - 8w_{2,2} &= 0, \\ 2w_{1,1} + (\lambda_3 - 8)w_{2,2} - 8w_{2,1} + (20 - 2\lambda_3)w_{2,2} &= 0. \end{aligned} \quad (5.294)$$

For the antisymmetric deflection of the plate (the symmetric deflection along the X -direction gives a higher buckling load),

$$w_{1,1} = w_{1,2} = w, \quad w_{2,1} = w_{2,2} = -w. \quad (5.295)$$

In view of Eqs. (5.295), Eqs. (5.294) reduce to a single equation which gives $\lambda_3 = 6$. Therefore,

$$(N_x)_{cr} = 54 \frac{D}{a^2} = (5.47) \pi^2 \frac{D}{a^2}. \quad (5.296)$$

For $n = 4$, the finite difference mesh for the plate is as shown in Fig. 5.23. For such a mesh,

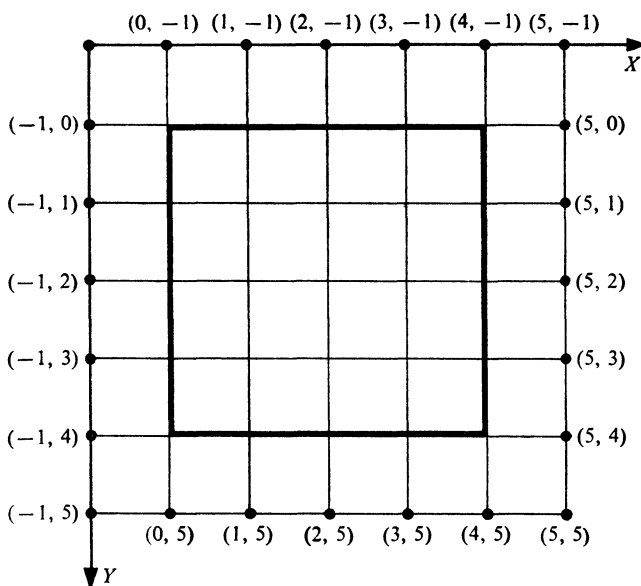


Fig. 5.23 Finite Difference Mesh for Plate ($n = 4$).

$$w_{1,1} = -w_{3,1}, \quad w_{1,2} = -w_{3,2}, \quad w_{1,3} = -w_{3,3}, \quad w_{2,1} = w_{2,3}.$$

Following the same procedure as for $n = 3$, the lowest root we get is $\lambda_4 = 3.82$. Hence,

$$(N_x)_{cr} = (61.1) \frac{D}{a^2} = (6.19) \pi^2 \frac{D}{a^2}. \quad (5.297)$$

The number of meshes can be increased depending on the desired accuracy of $(N_x)_{cr}$.

EXAMPLE 5.18 (Buckling of square plate under biaxial loading) Consider a square plate simply-supported on all sides and subjected to a biaxial loading, as shown in Fig. 5.24.

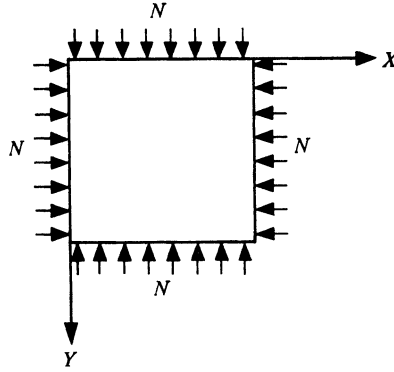


Fig. 5.24 Square Plate under Biaxial Compressive Loading.

Following Eq. (5.27), the governing equation for plate buckling we have is

$$\nabla^4 w + \frac{N}{D} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0. \quad (5.298)$$

The boundary conditions are

$$\begin{aligned} w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad (x = 0, a), \\ w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad (y = 0, a). \end{aligned} \quad (5.299)$$

In terms of the nondimensional parameters ξ and η , we can write the governing equation as

$$\begin{aligned} w_{i+2,j} - 8w_{i+1,j} + 20w_{i,j} - 8w_{i-1,j} + w_{i-2,j} + 2w_{i+1,j+1} + 2w_{i+1,j-1} \\ - 8w_{i,j+1} - 8w_{i,j-1} + w_{i,j+2} + w_{i,j-2} + 2w_{i-1,j+1} + 2w_{i-1,j-1} \\ + \lambda_n(w_{i+1,j} - 4w_{i,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) = 0. \end{aligned} \quad (5.300)$$

Expressing the boundary conditions (5.299) in terms of the differences, we have

$$\begin{aligned} w_{i,j} &= 0, & w_{i+1,j} &= -w_{i-1,j} & (i = 0, n, j = 0, 1, 2, \dots, n), \\ w_{i,j} &= 0, & w_{i,j+1} &= -w_{i,j-1} & (i = 0, 1, 2, \dots, n, j = 0, n). \end{aligned} \quad (5.301)$$

For $n = 3$, the finite difference mesh is the same as that obtained in Example 5.17. Writing the difference equations for the interior points (1, 1), (2, 1), (1, 2), and (2, 2), we get the homogeneous algebraic equations

$$\begin{aligned} w_{3,1} - 8w_{2,1} + 20w_{1,1} - 8w_{0,1} + w_{-1,1} + 2w_{2,2} + 2w_{2,0} - 8w_{1,2} - 8w_{1,0} + w_{1,3} \\ + w_{1,-1} + 2w_{0,2} + 2w_{0,0} + \lambda_3(w_{2,1} - 4w_{1,1} + w_{0,1} + w_{1,2} + w_{1,0}) &= 0, \\ w_{3,2} - 8w_{2,2} + 20w_{1,2} - 8w_{0,2} + w_{-1,2} + 2w_{2,3} + 2w_{2,1} - 8w_{1,3} - 8w_{1,1} + w_{1,4} \\ + w_{1,0} + 2w_{0,2} + 2w_{0,1} + \lambda_3(w_{2,2} - 4w_{1,2} + w_{0,2} + w_{1,3} + w_{1,1}) &= 0, \\ w_{4,1} - 8w_{3,1} + 20w_{2,1} - 8w_{1,1} + w_{0,1} + 2w_{3,2} + 2w_{3,0} - 8w_{2,2} - 8w_{2,0} + w_{2,3} \\ + w_{2,-1} + 2w_{1,2} + 2w_{1,0} + \lambda_3(w_{3,1} - 4w_{2,1} + w_{1,1} + w_{2,2} + w_{2,0}) &= 0, \\ w_{4,2} - 8w_{3,2} + 20w_{2,2} - 8w_{1,2} + w_{0,2} + 2w_{3,3} + 2w_{3,1} - 8w_{2,3} - 8w_{2,1} + w_{2,4} \\ + w_{2,0} + 2w_{1,3} + 2w_{1,1} + \lambda_3(w_{3,2} - 4w_{2,2} + w_{1,2} + w_{2,3} + w_{2,1}) &= 0. \end{aligned}$$

Incorporating the boundary values, we see the foregoing equations reduce to

$$\begin{aligned} (18 - 4\lambda_3)w_{1,1} + (\lambda_3 - 8)w_{1,7} + (\lambda_3 - 8)w_{2,1} + 2w_{2,2} &= 0, \\ (\lambda_3 - 8)w_{1,1} + (18 - 4\lambda_3)w_{1,2} + 2w_{2,1} + (\lambda_3 - 8)w_{2,2} &= 0, \\ (\lambda_3 - 8)w_{1,1} + 2w_{1,2} + (18 - 4\lambda_3)w_{2,1} + (\lambda_3 - 8)w_{2,2} &= 0, \\ 2w_{1,1} + (\lambda_3 - 8)w_{1,2} + (\lambda_3 - 8)w_{2,1} + (18 - 4\lambda_3)w_{2,2} &= 0. \end{aligned}$$

From the symmetry of the deflection, we have

$$w_{1,1} = w_{1,2} = w_{2,1} = w_{2,2} = w.$$

This results in only one independent equation, namely,

$$(4 - 2\lambda_3)w = 0. \quad (5.302)$$

This gives $\lambda_3 = 2$. Therefore,

$$N_{cr} = (18) \frac{D}{a^2}. \quad (5.303)$$

If we compare this value with the value (i.e., $2\pi^2 D/a^2$) obtained in Example 5.8 by using the energy approach, we find that the two differ by 10 per cent. Following a procedure similar to that for $n = 3$, we get, for $n = 4$,

$$N_{cr} = (18.75) \frac{D}{a^2}. \quad (5.304)$$

EXAMPLE 5.19 (Buckling of tapered rectangular plate compressed axially)

(i) *All sides simply-supported* Consider a rectangular plate as shown in Fig. 5.25. Assume that the plate is tapering along the X -direction and that its thickness t is

$$t = t_0(1 + \alpha \frac{x}{a})^{1/3}, \quad (5.305)$$

where α is the taper parameter. The equation governing the plate buckling is (5.181), that is,

$$D \nabla^4 w + 2 \frac{dD}{dx} \frac{\partial}{\partial x} (\nabla^2 w) + \frac{d^2 D}{dx^2} (\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}) + N_x \frac{\partial^2 w}{\partial x^2} = 0,$$

where the flexural rigidity D is

$$D = \frac{Et_0^3(1 + \alpha \frac{x}{a})}{12(1 - \nu^2)}. \quad (5.306)$$

Since D is a linear function of x , the third term in Eq. (5.181) vanishes. In terms of the

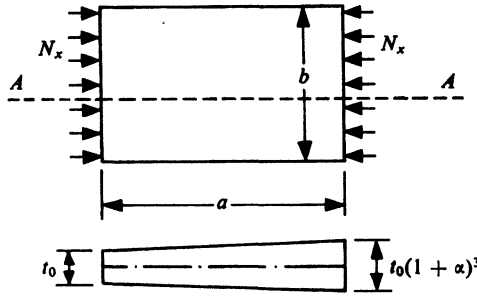


Fig. 5.25 Tapered Rectangular Plate under Axial Compression.

nondimensional parameters ξ and η , the governing equation now reduces to

$$(1 + \alpha \xi) (\frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4}) + 2\alpha (\frac{\partial^3 w}{\partial \xi^3} + p^2 \frac{\partial^3 w}{\partial \xi \partial \eta^2}) + N_x \frac{a^2}{D_0} \frac{\partial^2 w}{\partial \xi^2} = 0, \quad (5.307)$$

where

$$D_0 = \frac{Et_0^3}{12(1 - \nu^2)}.$$

The difference equations for the second, fourth, and cross derivatives are (5.288a), (5.288b), and (5.288e). The difference equations for the other derivatives required in Eq. (5.307) are

$$\frac{\partial^3 w}{\partial x^3} = (w_{i+2,j} - 2w_{i+1,j} + 2w_{i-1,j} - w_{i-2,j}) / (2h^3), \quad (5.308)$$

$$\frac{d}{dx}(\frac{d^2w}{dy^2}) = (w_{i+1,j+1} - w_{i-1,j+1} - 2w_{i+1,j} + 2w_{i-1,j} + w_{i+1,j-1} - w_{i-1,j-1})/(2h^3).$$

(5.309)

Substituting in Eq. (5.307) the differences corresponding to the various derivatives, we get

$$\begin{aligned} & (1 + \alpha\xi_i)[w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j} + 2p^2\{w_{i+1,j+1} - 2w_{i+1,j} \\ & \quad + w_{i+1,j-1} - 2(w_{i,j+1} - 2w_{i,j} + w_{i,j-1}) + w_{i-1,j+1} - 2w_{i-1,j} + w_{i-1,j-1}\} \\ & \quad + p^4(w_{i,j+2} - 4w_{i,j+1} + 6w_{i,j} - 4w_{i,j-1} + w_{i,j-2})] + \alpha h[w_{i+2,j} - 2w_{i+1,j} \\ & \quad + 2w_{i-1,j} - w_{i-2,j} + p^2(w_{i+1,j+1} - 2w_{i+1,j} + w_{i+1,j-1} - 2w_{i-1,j} - w_{i-1,j+1} \\ & \quad - w_{i-1,j-1})] + N_x \frac{a^2}{D_0} h^2 (w_{i+1,j} - 2w_{i,j} + w_{i-1,j}) = 0. \end{aligned}$$

(5.310)

The boundary conditions in terms of the differences are given by (5.301). It can be easily seen that Eq. (5.310) is more complicated than the one for a uniform plate. Therefore, a computer is required to carry out the calculations. In Table 5.4, we have given the buckling load N_x so obtained with varying internal points in a square plate with $\alpha = 1.0$. It can be seen from this table that the solution converges as the number of mesh points is increased.

Table 5.4 Variation of N_x with Internal Mesh Points ($\alpha = 1$ and $p = 1$)

Number of points	4	25	36	49
N_x	51.642	55.368	55.746	55.994

(ii) *Loaded edges simply-supported and other two edges clamped* Here too, the governing equation in the finite difference form is as given by (5.310). However, the boundary conditions are

$$\begin{aligned} w_{i,j} &= 0, \quad w_{i+1,j} = -w_{i-1,j} & (i = 0, n, j = 0, 1, 2, \dots, n), \\ w_{i,j} &= 0, \quad w_{i,j+1} = w_{i,j-1} & (i = 1, 2, \dots, n, j = 0, n). \end{aligned}$$

(5.311)

Here again, a computer has to be used for the calculations. The buckling load so obtained for a square plate with $\alpha = 1$ for different internal points is shown in Table 5.5.

Table 5.5 Variation of N_x with Internal Mesh Points ($\alpha = 1$ and $p = 1$)

Number of points	4	9	16	25	36	49	64
N_x	72.339	80.257	85.958	89.836	92.505	94.399	95.785

5.8.2 Finite Element Technique

The finite element technique (see Section 1.9.3) can also be applied to plates. A plate, just as a column, too can be discretized into a large number of elements, each connected to the other at the nodal points. A column, as noted earlier, is a one-dimensional structure, and when analyzing it, only beam elements are considered. On the other hand, in a plate, which is a two-dimensional structure, a variety of elements, e.g., triangular, rectangular, trapezoidal, and sectorial, are possible. The choice of element depends on the plate shape and boundary. From any of these simple elements (edges being straight), higher order elements can be obtained by increasing the number of nodes. A question that naturally arises at this stage is whether there is any economic or other advantage in doing this. Although, it is difficult to categorically answer it, in general, as the order of an element is increased, the number of unknowns can be reduced for a given accuracy of the solution. Consequently, the time needed for solving the resulting equations would reduce but that required for element formulation would increase. Economically, however, it is more advantageous if there is a reduction in the total computation time and data preparation effort. If a plate is flat with straight edges, it is then advisable to discretize it into any of the elements we have listed. If, on the other hand, a plate has arbitrary shape, it can then be discretized into curved isoparametric elements. The advantage in doing this is that the surface can be correctly mapped and also the shape functions for the geometry and displacements are the same. For a discussion on isoparametric elements, see Zienkiewicz [16, 17] and Desai and Abel [18].

In general, we can have two coordinate systems, namely, the *global coordinate system*, which is fixed in space and with respect to which the displacements, slopes, and moments are measured; and the *local coordinate system*, which changes from one element to another. The relationship between these two systems is expressed in terms of their direction cosines. The local coordinate system is usually preferred since it involves fewer number of variables. For example, consider an axial bar element lying in a plane; if the local coordinate axis lies along the axis of the bar, then only two axial displacements of the bar would be possible. However, for any axis not along the axis of the bar, there would be two displacements at each node, thus increasing the order of the resulting matrix.

There are two types of elements, namely, membrane elements and bending elements. If a plate is subjected only to inplane loads and can resist only such loads, membrane elements are then considered. On the other hand, if a plate can resist both transverse loads and bending moments, then bending elements are considered. Since, at buckling, bending moments develop and the plate resists them, we shall, in our discussion, consider only the bending elements.

A large number of plate bending elements have been developed and reported. In thin plate theory, the deformation of a plate is completely described by only the transverse deflection of the middle surface of the plate (w). Thus, the shape function should be so assumed that it satisfies not only the continuity of w but also of the derivatives of w with respect to the coordinate axes x and y between adjacent elements. According to the convergence requirements (Zienkiewicz [16]), such a shape function should represent constant strain state, implying that it must contain constant curvature states, i.e., $\partial^2 w / \partial x^2$ and $\partial^2 w / \partial y^2$, and constant twist, i.e., $\partial^2 w / (\partial x \partial y)$. Also, it must have geometric isotropy. Clearly then, for a

plate, it is much more difficult to choose a shape function satisfying all these requirements than for a one-dimensional structure, e.g., a column. To overcome this difficulty, especially in respect of triangular and quadrilateral elements, different investigators have developed various elements.

The shape function (i.e., displacement model) is a function of two variables (x, y). A displacement model which is continuous within an element and compatible between adjacent elements is termed *compatible* or *conforming*. A variety of shape functions is now available, some conformable and others nonconformable. It is rather difficult to generate the former category of elements. In here, we shall restrict ourselves to some typical nonconforming and conforming elements.

Triangular plate bending element (nonconforming)

In a triangular plate element, the degrees of freedom at each node are transverse displacement w , and rotations about the $\bar{X}$ - and $\bar{Y}$ -axis ($\partial w / \partial \bar{y}$ and $-\partial w / \partial \bar{x}$)—see Fig. 5.26. The

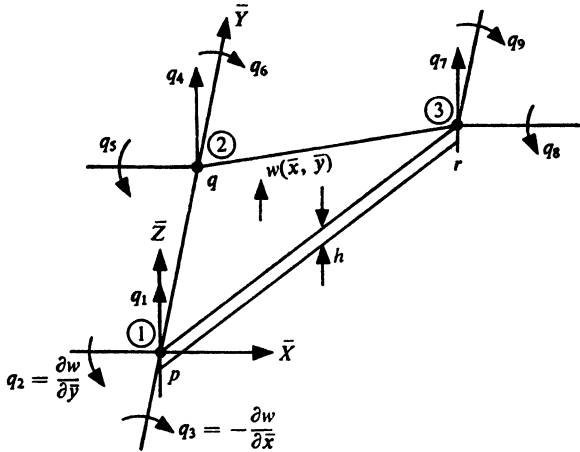


Fig. 5.26 Nodal Degrees of Freedom of Triangular Plate Bending Element.

negative sign in $\partial w / \partial \bar{x}$ indicates that if a positive displacement dw at a distance dx from node 1 is considered, then the rotation $\partial w / \partial \bar{x}$ about the $\bar{Y}$ -axis at node 1 will be opposite to the direction of q_3 . Since there are nine degrees of freedom for this element, the shape function must contain nine terms. The shape function can be taken as

$$w(\bar{x}, \bar{y}) = \alpha_1 + \alpha_2 \bar{x} + \alpha_3 \bar{y} + \alpha_4 \bar{x}^2 + \alpha_5 \bar{x} \bar{y} + \alpha_6 \bar{y}^2 + \alpha_7 \bar{x}^3 + \alpha_8 (\bar{x}^2 \bar{y} + \bar{x} \bar{y}^2) + \alpha_9 \bar{y}^3$$

$$= [\phi] \alpha, \quad (5.312)$$

where

$$[\phi] = [1 \quad \bar{x} \quad \bar{y} \quad \bar{x}^2 \quad \bar{x} \bar{y} \quad \bar{y}^2 \quad \bar{x}^3 \quad (\bar{x}^2 \bar{y} + \bar{x} \bar{y}^2) \quad \bar{y}^3],$$

$$\alpha^T = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_9].$$

The constants $\alpha_1, \alpha_2, \dots, \alpha_9$ are determined from the conditions

$$\begin{aligned} w(\bar{x}_1, \bar{y}_1) &= q_1, & \frac{\partial w}{\partial \bar{y}}(\bar{x}_1, \bar{y}_1) &= q_2, & -\frac{\partial w}{\partial \bar{x}}(\bar{x}_1, \bar{y}_1) &= q_3, \\ w(\bar{x}_2, \bar{y}_2) &= q_4, & \frac{\partial w}{\partial \bar{y}}(\bar{x}_2, \bar{y}_2) &= q_5, & -\frac{\partial w}{\partial \bar{x}}(\bar{x}_2, \bar{y}_2) &= q_6, \\ w(\bar{x}_3, \bar{y}_3) &= q_7, & \frac{\partial w}{\partial \bar{y}}(\bar{x}_3, \bar{y}_3) &= q_8, & -\frac{\partial w}{\partial \bar{x}}(\bar{x}_3, \bar{y}_3) &= q_9, \end{aligned} \quad (5.313)$$

where $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_2, \bar{y}_2)$, and $(\bar{x}_3, \bar{y}_3)$ are the coordinates of nodes 1, 2, and 3, respectively.

As noted earlier in this section, a local coordinate system is preferred. We therefore choose this system for the element. One such system (see Fig. 5.26) is to take, as the $\bar{Y}$ -axis, the line connecting nodes 1 and 2, and the $\bar{X}$ -axis as normal to the $\bar{Y}$ -axis in the direction of node 3 with the origin at node 1. Further, we assume that the local node numbers 1, 2, and 3 correspond to the nodes p, q , and r , respectively, in the global coordinate system.

Using relations (5.313), we can write Eq. (5.312) as

$$q = [A]\alpha, \quad (5.314)$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \bar{y}_2 & 0 & 0 & \bar{y}_2^2 & 0 & 0 & \bar{y}_2^3 \\ 0 & 0 & 1 & 0 & 0 & 2\bar{y}_2 & 0 & 0 & 3\bar{y}_2^2 \\ 0 & -1 & 0 & 0 & -\bar{y}_2 & 0 & 0 & -\bar{y}_2^2 & 0 \\ 1 & \bar{x}_3 & \bar{y}_3 & \bar{x}_3^2 & \bar{x}_3\bar{y}_3 & \bar{y}_3^2 & \bar{x}_3^3 & (\bar{x}_3^2\bar{y}_3 + \bar{x}_3\bar{y}_3^2) & \bar{y}_3^3 \\ 0 & 0 & 1 & 0 & \bar{x}_3 & 2\bar{y}_3 & 0 & (2\bar{x}_3\bar{y}_3 + \bar{x}_3^2) & 3\bar{y}_3^2 \\ 0 & -1 & 0 & -2\bar{x}_3 & -\bar{y}_3 & 0 & -3\bar{x}_3^2 & -(\bar{y}_3^2 + 2\bar{x}_3\bar{y}_3) & 0 \end{bmatrix}.$$

Substituting for α from Eq. (5.312) in Eq. (5.314), we have

$$\begin{aligned} w &= [\phi][A]^{-1}q \\ &= [N]q. \end{aligned} \quad (5.315)$$

Equation (5.315) expresses the displacement w at any point within the element in terms of the nodal displacements.

As already noted, the strain-displacement relation is given by Eq. (1.263a) or (1.263b), where the matrix $[B_\alpha]$ is

$$[B_\alpha] = -z \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix},$$

$$[B] = [B_a][A]^{-1}. \quad (5.316)$$

Also, the element stiffness matrix $[k^{(e)}]$ in the local coordinate system is given by Eq. (1.269), i.e.,

$$[k^{(e)}] = \int_{\text{vol}^{(e)}} [B]^T [X] [B] dv. \quad (1.269)$$

The material matrix $[X]$ is given by Eqs. (I.14) which, after rearrangement, become

$$[X] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \quad (5.317)$$

Substituting for $[B]$ and $[X]$ in Eq. (1.269), we have

$$[k^{(e)}] = ([A]^{-1})^T \left\{ \int_{A^{(e)}} dA \left(\int_{-h/2}^{h/2} [B_a]^T [X] [B] dz \right) \right\} [A]^{-1}. \quad (5.318)$$

It can be seen that the evaluation of element stiffness matrix involves the numerical determination of the inverse of the (9×9) matrix, i.e., (5.318). Finally, the element stiffness matrix in the global coordinate system (where the XY -plane is assumed to be the same as the local $\bar{X}\bar{Y}$ -plane) can be obtained from

$$[K^{(e)}] = [\lambda]^T [k^{(e)}] [\lambda], \quad (5.319)$$

where the transformation matrix $[\lambda]$ is given by

$$[\lambda] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{ox}^- & m_{ox}^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{oy}^- & m_{oy}^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{ox}^- & m_{ox}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{oy}^- & m_{oy}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{ox}^- & m_{ox}^- \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{oy}^- & m_{oy}^- \end{bmatrix} \quad (5.320)$$

with (l_{ox}^-, m_{ox}^-) and (l_{oy}^-, m_{oy}^-) representing the direction cosines of the lines ox and oy . Knowing $[K^{(e)}]$ and following the method suggested in Section 1.9.3, we can obtain the global elastic stiffness matrix $[K]$ of a structure.

Rectangular plate bending element (nonconforming)

Figure 5.27 shows a rectangular plate element with three degrees of freedom, namely, w , $\partial w / \partial \bar{y}$, and $-\partial w / \partial \bar{x}$, at each node. Thus, there are 12 degrees of freedom for this element.

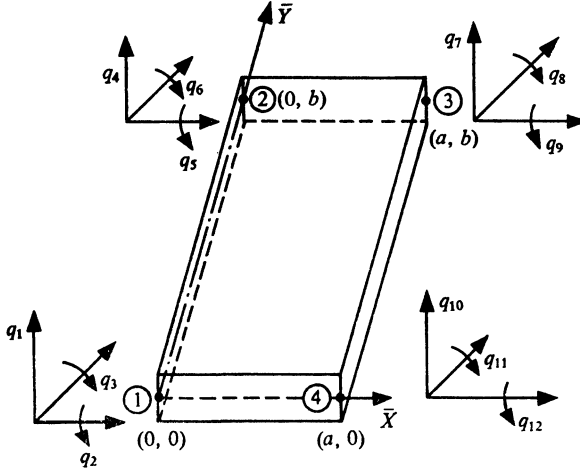


Fig. 5.27 Rectangular Plate Bending Element.

Therefore, the displacement model must contain 12 constants; it can be expressed as

$$w(\bar{x}, \bar{y}) = \alpha_1 + \alpha_2 \bar{x} + \alpha_3 \bar{y} + \alpha_4 \bar{x}^2 + \alpha_5 \bar{x} \bar{y} + \alpha_6 \bar{y}^2 + \alpha_7 \bar{x}^3 + \alpha_8 \bar{x}^2 \bar{y} + \alpha_9 \bar{x} \bar{y}^2 + \alpha_{10} \bar{y}^3 + \alpha_{11} \bar{x}^3 \bar{y} + \alpha_{12} \bar{x} \bar{y}^3, \quad (5.321)$$

that is,

$$w = [\phi] \alpha, \quad (5.322)$$

where

$$[\phi]^T = [1 \quad \bar{x} \quad \bar{y} \quad \bar{x}^2 \quad \bar{x} \bar{y} \quad \bar{y}^2 \quad \bar{x}^3 \quad \bar{x}^2 \bar{y} \quad \bar{x} \bar{y}^2 \quad \bar{y}^3 \quad \bar{x}^3 \bar{y} \quad \bar{x} \bar{y}^3],$$

$$\alpha^T = [\alpha_1, \alpha_2, \dots, \alpha_{12}]^T.$$

The constants $\alpha_1, \alpha_2, \dots, \alpha_{12}$ are determined from the conditions

$$w(\bar{x}_1, \bar{y}_1) = q_1, \quad \frac{\partial w}{\partial \bar{y}}(\bar{x}_1, \bar{y}_1) = q_2, \quad -\frac{\partial w}{\partial \bar{x}}(\bar{x}_1, \bar{y}_1) = q_3,$$

$$w(\bar{x}_2, \bar{y}_2) = q_4, \quad \frac{\partial w}{\partial \bar{y}}(\bar{x}_2, \bar{y}_2) = q_5, \quad -\frac{\partial w}{\partial \bar{x}}(\bar{x}_2, \bar{y}_2) = q_6,$$

$$w(\bar{x}_3, \bar{y}_3) = q_7, \quad \frac{\partial w}{\partial \bar{y}}(\bar{x}_3, \bar{y}_3) = q_8, \quad -\frac{\partial w}{\partial \bar{x}}(\bar{x}_3, \bar{y}_3) = q_9,$$

$$w(\bar{x}_4, \bar{y}_4) = q_{10}, \quad \frac{\partial w}{\partial \bar{y}}(\bar{x}_4, \bar{y}_4) = q_{11}, \quad -\frac{\partial w}{\partial \bar{x}}(\bar{x}_4, \bar{y}_4) = q_{12}.$$

Here, the matrix $[A]$, which relates the nodal degrees of freedom $(q_1, q_2, \dots, q_{12})$ to the generalized coordinates $(\alpha_1, \alpha_2, \dots, \alpha_{12})$, is given by

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & -b & 0 & 0 & 0 & -b^2 & 0 & 0 & -b^3 \\ 1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & ab^3 \\ 0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab^2 \\ 0 & -1 & 0 & -2a & -b & 0 & -3a^2 & -2ab & -b^2 & 0 & -3a^2b & -b^3 \\ 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\ 0 & -1 & 0 & -2a & 0 & 0 & -3a^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5.323)$$

The strain-displacement relation is given by Eq. (1.263a) or (1.263b), where the matrix $[B_a]$ is

$$[B_a] = -z \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix},$$

$$[B] = [B_a][A^{-1}]. \quad (5.324)$$

The material matrix $[X]$ is given by Eq. (5.317). Substituting the expressions for $[B]$ and $[X]$ in Eq. (5.318), we can obtain the element stiffness matrix in the local coordinate system.

If the global XY -plane coincides with the local $\bar{X}\bar{Y}$ -plane, the stiffness matrix of the element in the global coordinate system is given by

$$[K^{(e)}] = [\lambda]^T [k^{(e)}] [\lambda], \quad (5.325)$$

where

$$[\lambda] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{ox} & m_{ox} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{oy} & m_{oy} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{ox} & m_{ox} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{oy} & m_{oy} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{ox} & m_{ox} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{oy} & m_{oy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{ox} & m_{ox} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{oy} & m_{oy} \end{bmatrix}. \quad (5.326)$$

Triangular plate bending element (conforming)

The triangular bending element we have considered violates the continuity of slopes requirement along the common boundaries of neighbouring elements. This means that the boundaries of such elements may have discontinuities of slope. The triangular element developed by Cowper et al [19] satisfies the continuity of slopes requirement between adjacent elements as also the other convergence requirements we outlined earlier in this section. Their element has six degrees of freedom, namely, w , $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, and $\partial^2 w/(\partial x \partial y)$, at each node. Thus, there are 18 degrees of freedom. In this situation, the displacement model is assumed to be a quintic polynomial in $\bar{x}$ and $\bar{y}$. This is written as

$$\begin{aligned} w(\bar{x}, \bar{y}) = & \alpha_1 + \alpha_2 \bar{x} + \alpha_3 \bar{y} + \alpha_4 \bar{x}^2 + \alpha_5 \bar{x} \bar{y} + \alpha_6 \bar{y}^2 + \alpha_7 \bar{x}^3 + \alpha_8 \bar{x}^2 \bar{y} + \alpha_9 \bar{x} \bar{y}^2 + \alpha_{10} \bar{y}^3 \\ & + \alpha_{11} \bar{x}^4 + \alpha_{12} \bar{x}^3 \bar{y} + \alpha_{13} \bar{x}^2 \bar{y}^2 + \alpha_{14} \bar{x} \bar{y}^3 + \alpha_{15} \bar{y}^4 + \alpha_{16} \bar{x}^5 + \alpha_{17} \bar{x}^3 \bar{y}^2 \\ & + \alpha_{18} \bar{x}^2 \bar{y}^3 + \alpha_{19} \bar{x} \bar{y}^4 + \alpha_{20} \bar{y}^5 + \alpha_{21} \bar{x}^4 \bar{y}. \end{aligned} \quad (5.327)$$

In order to evaluate the general coordinates $\alpha_1, \alpha_2, \dots, \alpha_{21}$, we need to have 21 conditions. Of these, 18 conditions can be obtained from the 18 nodal degrees of freedom. Thus, we still need 3 more conditions. These conditions are that the variation of the slope normal to an edge be a cubic function of the edge-wise coordinate. Such a cubic is uniquely defined by two quantities (namely, the normal slope and the twist at the ends of an edge) and ensures the continuity of displacement and slopes between adjacent elements.

To obtain all the generalized coordinates, the local and global coordinate systems Cowper et al [19] chose are as shown in Fig. 5.28. Here, the dimensions a , b , and c in terms

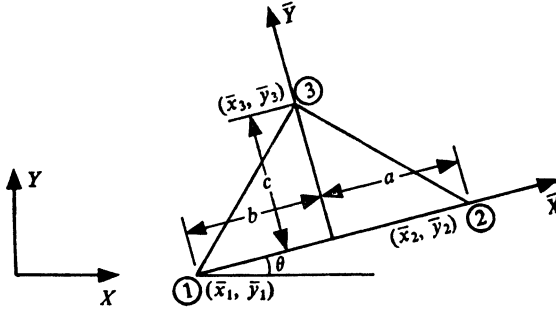


Fig. 5.28 Coordinate System for Bending Element.

of the global system are

$$\begin{aligned} a &= \{(X_2 - X_3)(X_2 - X_1) + (Y_2 - Y_3)(Y_2 - Y_1)\}/\beta, \\ b &= \{(X_3 - X_1)(X_2 - X_1) + (Y_3 - Y_1)(Y_2 - Y_1)\}/\beta, \\ c &= \{(X_2 - X_1)(Y_3 - Y_1) - (X_3 - X_1)(Y_2 - Y_1)\}/\beta, \end{aligned} \quad (5.328)$$

where

$$\beta = \{(X_2 - X_1)^2 + (Y_2 - Y_1)^2\}^{1/2}.$$

If we want to enforce the condition that the normal slope $(\partial w / \partial \bar{y})$ be a cubic equation in $\bar{x}$ along the edge $\bar{y} = 0$ (i.e., edge 1-2), we have to set $\alpha_{21} = 0$ in Eq. (5.327) as otherwise the power of $\bar{x}$ along this edge will be a quartic. We can then write Eq. (5.327) as

$$w = [\phi]\alpha,$$

where

$$[\phi] = [1 \quad \bar{x} \quad \bar{y} \quad \bar{x}^2 \dots \bar{y}^5],$$

$$\alpha^T = [\alpha_1, \alpha_2, \dots, \alpha_{20}].$$

The procedure to obtain the other two slope conditions is quite involved, and hence is omitted. These conditions are (Bogner et al [20])

$$5b^4c\alpha_{16} + (3b^2c^3 - 2b^4c)\alpha_{17} + (2bc^4 - 3b^3c^2)\alpha_{18} + (c^5 - 4b^2c^3)\alpha_{19} - 5bc^4\alpha_{20} = 0,$$

$$5a^4c\alpha_{16} + (3a^2c^3 - 2a^4c)\alpha_{17} + (3a^3c^2 - 2ac^4)\alpha_{18} + (c^5 - 4a^2c^3)\alpha_{19} + 5ac^4\alpha_{20} = 0$$

and are along the edges 1-3 and 2-3, respectively (see Fig. 5.28). Using these equations, we can generate the matrix $[A]$. The strain-displacement relation is given by Eq. (1.263a) or (1.263b), where the matrix $[B_a]$ is

$$[B_a] = -z \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 6x & 2y & 0 & 0 & 12x^2 & 6xy & 2y^2 & 0 & 0 & 20x^3 & 6xy^2 & 2y^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x & 6y & 0 & 0 & 2x^2 & 6xy & 12y^2 & 0 & 2x^3 & 6x^2y & 12xy^2 & 20y^3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2x & 2y & 0 & 0 & 3x^2 & 4xy & 3y^2 & 0 & 0 & 6x^2y & 6xy^2 & 4y^3 & 0 \end{bmatrix}, \quad (5.329)$$

$$[B] = [B_a][A^{-1}].$$

The material matrix $[X]$ is given by Eq. (5.317). Knowing $[B]$ and $[X]$, we can obtain the element stiffness matrix in the local coordinate system. The transformation matrix $[\lambda]$ is given by

$$[\lambda] = \begin{bmatrix} [\bar{\lambda}] & [0] & [0] \\ [0] & [\bar{\lambda}] & [0] \\ [0] & [0] & [\bar{\lambda}] \end{bmatrix}, \quad (5.330)$$

where

$$[\bar{\lambda}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 & 0 & 0 \\ 0 & -\sin \theta & \cos \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2 \theta & 2 \sin \theta \cos \theta & \sin^2 \theta \\ 0 & 0 & 0 & 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta & \sin \theta \cos \theta \\ 0 & 0 & 0 & \sin^2 \theta & -2 \sin \theta \cos \theta & \cos^2 \theta \end{bmatrix},$$

$[0] = \text{null matrix of order } 6 \times 6.$

Here, θ denotes the angle between the $\bar{X}$ - and X -axis. The global stiffness matrix is given by

$$[K^{(e)}] = [\lambda]^T [k^{(e)}] [\lambda]. \quad (5.331)$$

Rectangular plate bending element (conforming)

The rectangular bending element proposed by Bogner et al [20] also satisfies the continuity of slopes requirement between adjacent elements and the other convergence criteria. In their element, there are four degrees of freedom, namely, w , $\partial w / \partial \bar{x}$, $\partial w / \partial \bar{y}$, and $\partial^2 w / (\partial \bar{x} \partial \bar{y})$, at each node. The displacement model $w(x, y)$ they chose is

$$w(\bar{x}, \bar{y}) = \sum_{i=1}^2 \sum_{j=1}^2 \{ H_{0i}^{(1)}(\bar{x}) + H_{0i}^{(1)}(\bar{y}) w_{ij} + H_{1i}^{(1)}(\bar{x}) H_{0j}^{(1)}(\bar{y}) \left(\frac{\partial w}{\partial \bar{y}} \right)_{ij} + H_{0i}^{(1)}(\bar{x}) H_{1j}^{(1)}(\bar{y}) \left(\frac{\partial w}{\partial \bar{x}} \right)_{ij} + H_{1i}^{(1)}(\bar{x}) H_{1j}^{(1)}(\bar{y}) \left(\frac{\partial^2 w}{\partial \bar{x} \partial \bar{y}} \right)_{ij} \}, \quad (5.332)$$

where $H_{01}, H_{02}, \dots$ are the first order Hermitian polynomials and are defined as (Zienkiewicz [16])

$$H_{01}^{(1)}(\bar{x}) = \frac{1}{a^3} (2\bar{x}^3 - 3a\bar{x}^2 + a^3),$$

$$H_{02}^{(1)}(\bar{x}) = -\frac{1}{a^3} (2\bar{x}^3 - 3a\bar{x}^2),$$

$$H_{11}^{(1)}(\bar{x}) = \frac{1}{a^2} (\bar{x}^3 - 2a\bar{x}^2 + a^2\bar{x}),$$

$$H_{12}^{(1)}(\bar{x}) = \frac{1}{a^2} (\bar{x}^3 - a\bar{x}^2),$$

and $H_{ij}^{(1)}(\bar{y})$ can be obtained by replacing $\bar{x}$ by $\bar{y}$ and a by b . The subscripts ij correspond to the nodes shown in Fig. 5.29. Once $w(\bar{x}, \bar{y})$ is assumed, the matrices $[B]$ and $[X]$, and hence the elastic stiffness matrix in the local coordinate system, can be obtained.

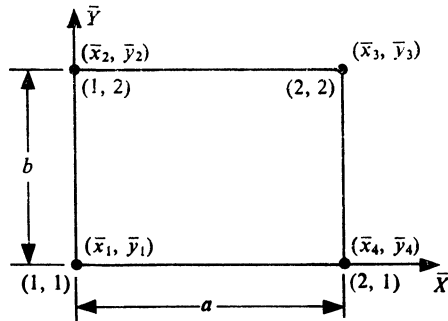


Fig. 5.29 Coordinate System for Plate Element.

To obtain the buckling load, we need to calculate the geometric stiffness matrix (see Section 1.9.3).

In a situation where all the inplane stresses are acting, the element geometric stiffness matrix $[g_e]$ is given by Eq. (1.276), i.e.,

$$[g_e] = \sigma_x \int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial x} \right) dv + \sigma_y \int_{\text{vol}} \left(\frac{\partial \phi}{\partial y} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv + 2\tau_{xy} \int_{\text{vol}} \left(\frac{\partial \phi}{\partial x} \right)^T \left(\frac{\partial \phi}{\partial y} \right) dv, \quad (5.333)$$

$$[g] = [A^{-1}]^T [g_e] [A^{-1}].$$

After we know $[g]$, we can obtain the global geometric stiffness matrix $[G]$. As in columns, so too here, we can use either the consistent or nonconsistent approach to obtain the buckling load.

EXAMPLE 5.20 (Buckling of simply-supported square plate) Consider a square plate simply-supported on all sides and compressed along the X -axis, as shown in Fig. 5.30. Since it has a regular shape, it can be divided into a number of square elements. For our analysis, we shall treat these elements as of the nonconforming type because the number of degrees of freedom in such a type is less than that in a conforming type of element. Figure 5.31 shows a typical square element together with its coordinates.

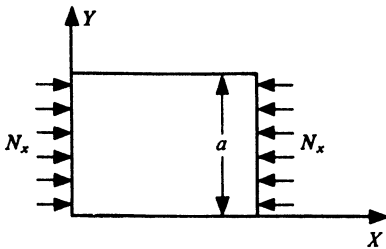


Fig. 5.30 Square Plate under Compressive Loading.

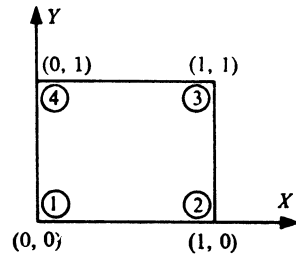


Fig. 5.31 Typical Square Element.

The degrees of freedom at each node of the element are (w, θ_x, θ_y) , where $\theta_x = \partial w / \partial y$ and $\theta_y = -\partial w / \partial x$. For such an element, the displacement function is given by Eq. (5.321). The nodal degrees of freedom are related to the generalized coordinates as

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ \partial w / \partial y \\ -\partial w / \partial x \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -2xy & -y^2 & 0 & -3x^2y & -y^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{12} \end{Bmatrix}. \quad (5.334)$$

The matrices $[A]$, $[B_e]$, and $[X]$ are given by (5.323), (5.324), and (5.317), respectively. Substituting these equations in Eq. (5.318) and integrating, we get the elastic stiffness

matrix for the element $[k^e]$ as

$$[k^e] = \begin{bmatrix} 0 & & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & 4 & & & & & & & & \\ 0 & 0 & 0 & 0 & 1.4 & & & & & & & \\ 0 & 0 & 0 & 1.2 & 0 & 4 & & & & & & \\ 0 & 0 & 0 & 6 & 0 & 1.8 & 12 & & & & & \\ 0 & 0 & 0 & 2 & 1.4 & 0.6 & 3 & 3.2 & & & & \\ 0 & 0 & 0 & 0.6 & 1.4 & 2 & 1.2 & 1.7 & 4.2 & & & \\ 0 & 0 & 0 & 1.8 & 0 & 6 & 2.7 & 1.2 & 3.0 & 12 & & \\ 0 & 0 & 0 & 3 & 1.4 & 0.9 & 6 & 2.7 & 2 & 1.8 & 6.5 & \\ 0 & 0 & 0 & 0.9 & 1.4 & 3 & 1.8 & 2.0 & 4.1 & 6.0 & 2.6 & 6.5 \end{bmatrix} \quad (5.335)$$

The geometric stiffness matrix $[g]$ is derived by using Eqs. (1.277) and (5.333) after dropping the contributions from N_y and N_{xy} . The term $\partial\theta/\partial x$ corresponds to $\partial w/\partial x$ which is given by

$$\frac{\partial w}{\partial x} = [0 \quad 1 \quad 0 \quad 2x \quad y \quad 0 \quad 3x^2 \quad 2xy \quad y^2 \quad 0 \quad 3x^2y \quad y^3] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{12} \end{Bmatrix} \quad (5.336)$$

Making use of Eq. (1.277), we get $[g_a]$ as

$$[g_a] = \begin{bmatrix} 0 & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 1 & 0 & \frac{4}{3} & & & & & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & \frac{3}{2} & & & & & \\ 0 & \frac{1}{2} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & \frac{3}{4} & \frac{4}{9} & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & \frac{1}{2} & 0 & \frac{3}{4} & \frac{1}{3} & 0 & -\frac{9}{10} & \frac{1}{2} & \frac{1}{4} & 0 & \frac{3}{5} & \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{5} & 0 & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & 0 & \frac{1}{5} & \frac{1}{7} \end{bmatrix} \quad (5.337)$$

Since the plate structure is symmetric, we need consider only one quarter of the plate to obtain the global elastic stiffness and geometric stiffness matrices.

In Table 5.6, we show the results we have obtained, using a computer, by considering varying number of elements in the plate.

Table 5.6 Values of $[N_x a^2/(\pi^2 D)]$ for Simply-Supported Square Plate

Number of elements	Quarter plate	Full plate
2×2	3.714	3.150
3×3	3.869	3.575
4×4	3.927	3.749
5×5	3.9536	3.8358

For more details on the application of the finite element technique to plate buckling problems, see Kapur and Hartz [21] and Anderson et al [22].

PROBLEMS

5.1 Obtain the buckling load of a rectangular plate of uniform thickness, simply-supported along two opposite sides and uniformly compressed (see Fig. 5.32) for each of the following sets of conditions:

The edges $x = 0, a$ are simply-supported and

- (i) the edges $y = \pm b/2$ are simply-supported;
- (ii) the edges $y = \pm b/2$ are clamped;
- (iii) the edge $y = +b/2$ is simply-supported, the edge $y = -b/2$ is clamped;
- (iv) the edge $y = +b/2$ is simply-supported, the edge $y = -b/2$ is free.

Assume the plate to be of aluminium.

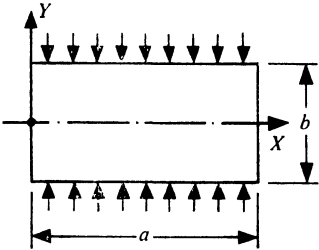


Fig. 5.32 Problem 5.1.

5.2 Obtain an expression for the buckling load of a rectangular plate, simply-supported along the edges $x = 0, a$ and supported elastically by beams of area of cross-section A along the edges $y = \pm b/2$ (see Fig. 5.33).

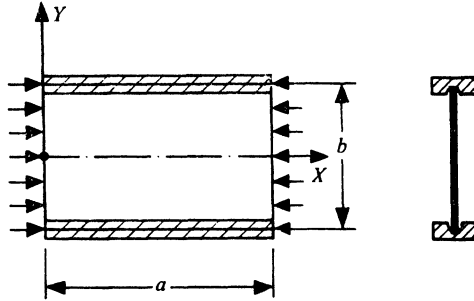


Fig. 5.33 Problem 5.2.

5.3 Obtain the buckling load of a rectangular plate, simply-supported along the edges $x = 0, a$ and free along the edges $y = 0, b$ and subjected to an inplane load N_x per unit length (see Fig. 5.34), given by

$$N_x = N_1 + (N_2 - N_1) \sin \frac{\pi y}{b}.$$

Also, plot the buckling coefficient k versus (N_1/N_2) .

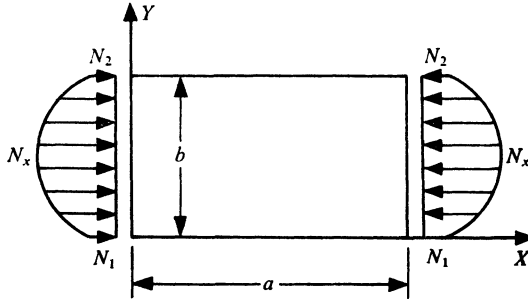


Fig. 5.34 Problem 5.3.

5.4 A rectangular plate, clamped on all sides, is stiffened by two transverse stiffeners, as shown in Fig. 5.35. The stiffeners are of the same material as the plate. Further, I_1 and I_2

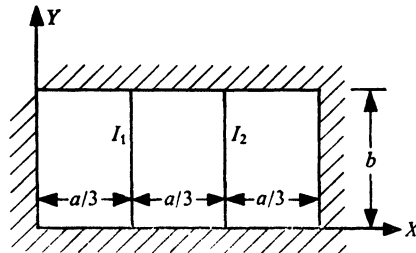


Fig. 5.35 Problem 5.4.

are the second moment of area of the stiffeners. Obtain an expression for the buckling load. The plate is uniformly compressed along the edges $x = 0$ and $x = a$.

5.5 Solve Problem 5.4, assuming that the compressive load is applied along the edges $y = 0, b$ and on the stiffeners. Also assume that the areas of the stiffeners are A_1 and A_2 .

5.6 A simply-supported rectangular plate is subjected to end loads along the edges $x = 0, a$ (see Fig. 5.36). Obtain an expression for the critical load in terms of α and the material parameters. Given $N(y) = N_0[1 - \alpha(y/b)]$.

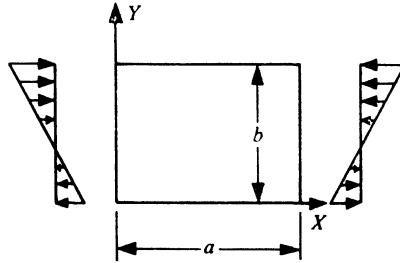


Fig. 5.36 Problem 5.6.

5.7 A $0.1 \text{ m} \times 0.25 \text{ m} \times 0.001 \text{ m}$ aluminium alloy ($E = 58.86 \text{ GPa}$, $\nu = 0.3$) skin panel of an aircraft wing is subjected to a minimum longitudinal compressive stress σ_x and a shear flow q . Draw the interaction curve for the buckling of the panel. Assume the panel to be simply-supported on all sides.

5.8 Repeat Problem 5.7, with compressive edges simply-supported and the other two edges clamped.

5.9 Determine the critical load of a simply-supported rectangular plate compressed in two perpendicular directions by a uniformly distributed load, as shown in Fig. 5.37. Obtain the solution by using the (i) Rayleigh-Ritz method and (ii) Galerkin technique.

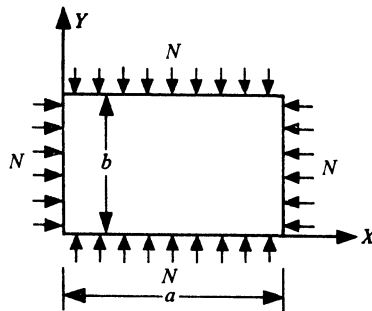


Fig. 5.37 Problem 5.9.

5.10 Use the Rayleigh-Ritz method to determine the buckling load for a square plate simply-supported on all sides and loaded along the X -direction by a linearly varying load (Fig. 5.38).

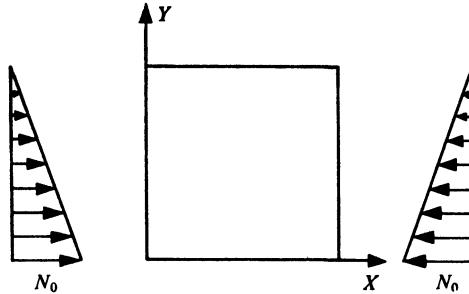


Fig. 5.38 Problem 5.10.

5.11 Solve Problem 5.9 by using triangular and rectangular nonconforming type of elements. Compare the results so obtained with those arrived at in Problem 5.9.

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6

Buckling of Composite Plates

6.1 INTRODUCTION

In Chapter 5, we discussed the buckling of two-dimensional structures, namely, plates—both stiffened and unstiffened—, subjected to inplane uniaxial and biaxial loads. In doing so, we assumed that the plate material is isotropic, that is, the mechanical properties of the material are the same in all the directions. The mechanical properties of a material can be changed only by alloying or by heat treatment.

A composite material consists of two or more materials and offers a significant weight saving in a structure in view of its high strength to weight and high stiffness to weight ratios. Further, in a fibrous composite, the mechanical properties can be varied as required by suitably orienting the fibres. In such a material, the fibres are the main load bearing members, and the matrix, which has low modulus and high elongation, provides the necessary flexibility and also keeps the fibres in position and protects them from the environment.

A lamina, in general, is a thin sheet with fibres oriented in some direction. Such a sheet can be characterized as two-dimensional, with orthogonal material properties. It is, however, not capable of carrying any load. Hence, for practical purposes, a structure consisting of several laminae is used. If the fibres in the laminae are oriented along the same direction, then the structure behaves like an orthotropic laminate. If, however, the fibre orientations of the laminae are different, the laminate behaviour is anisotropic. The buckling analysis of such a plate is more involved than that of an isotropic plate.

In this chapter, we shall analyze the composite plates subjected to both uniaxial and biaxial loads. In doing this, we shall assume that the deflection is small and the thickness of the plate is smaller than the other two characteristic dimensions.

6.2 GOVERNING EQUATION

A laminated plate is made by using a lamina as the building block. Its stiffness is obtained from the properties of the constituent laminae. To do this, we should know the orientations of the principal material directions of the laminae with respect to the laminate axes. Therefore, a knowledge of the variation of stress and strain through the laminate thickness is necessary.

We shall make the following assumptions regarding the behaviour of a laminate:

- (i) It is made up of perfectly bonded laminae.
- (ii) The bonds are infinitesimally thin and no lamina can slip relative to the other.

This implies that the displacements are continuous across the lamina boundaries. As a result, the laminate behaves like a lamina with special properties.

(iii) After buckling, a line originally straight and perpendicular to the middle surface of the laminate remains straight and perpendicular to the middle surface, that is, $\gamma_{xz} = \gamma_{yz} = 0$.

(iv) The strain perpendicular to the middle surface of the laminate is ignored.

Figure 6.1 shows the deformation of the plate plane in the XZ -plane. Let u , v , w

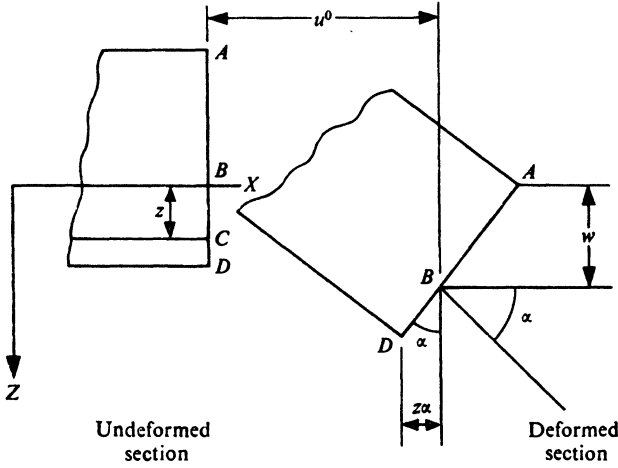


Fig. 6.1 Deformation in XZ -Plane.

represent the displacements along the X -, Y -, Z -direction and u^0 , v^0 the deformation of the laminate middle surface. The displacement along the X -axis of any point C at a distance z from the X -axis is

$$u = u^0 - \alpha z, \quad (6.1)$$

where α is the slope of the middle surface in the X -direction, that is,

$$\alpha = \frac{\partial w}{\partial x}. \quad (6.2)$$

Hence, the displacement u at any point z through the laminate thickness is

$$u = u^0 - z \frac{\partial w}{\partial x}. \quad (6.3)$$

Similarly, we have the displacement v along the Y -axis given by

$$v = v^0 - z \frac{\partial w}{\partial y}. \quad (6.4)$$

Using the strain-displacement relations

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x}, \\ \epsilon_y &= \frac{\partial v}{\partial y}, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\end{aligned}\tag{6.5}$$

we can write the strains in terms of middle surface displacements as

$$\begin{aligned}\epsilon_x &= \frac{\partial u^0}{\partial x} - z \frac{\partial^2 w}{\partial x^2}, \\ \epsilon_y &= \frac{\partial v^0}{\partial y} - z \frac{\partial^2 w}{\partial y^2}, \\ \gamma_{xy} &= \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y}.\end{aligned}\tag{6.6}$$

Equations (6.6) can be rewritten as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix},\tag{6.7}$$

where

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u^0}{\partial x} \\ \frac{\partial v^0}{\partial y} \\ \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \end{Bmatrix},\tag{6.8}$$

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}.\tag{6.9}$$

Substituting Eq. (6.7) in the stress-strain relations, i.e., (I.21)—see Appendix I—, we can express the stresses in the r -th layer in terms of the laminate middle surface strains and

curvatures as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_r = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_r \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \quad (6.10)$$

Since the reduced stiffness matrix $\bar{Q}_{ij}$ is different for each lamina, the stress variation through the laminate thickness need not be linear, even though the strain variation through the thickness is linear. Knowing the stresses in terms of the displacement, we can obtain the stress resultants N_x , N_y , N_{xy} , M_x , M_y , and M_{xy} from Eqs. (5.2). Thus, the forces and moments for an N -layer laminate can be defined as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_r dz = \sum_{r=1}^N \int_{z_{r-1}}^{z_r} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_r dz, \quad (6.11)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_r z dz = \sum_{r=1}^N \int_{z_{r-1}}^{z_r} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_r z dz, \quad (6.12)$$

where z_r and z_{r-1} are as defined in Fig. 6.2. Substituting for σ_x , σ_y , and τ_{xy} from Eq. (6.10)

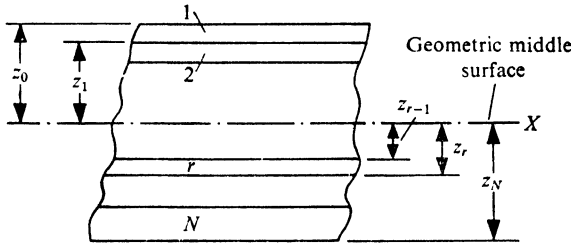


Fig. 6.2 z -coordinates of N -Layer Laminate.

in Eqs. (6.11) and (6.12) and integrating over the thickness of each layer and adding the results so obtained for N layers, we can write the stress resultants as (Jones [1])

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad (6.13)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad (6.14)$$

where

$$\begin{aligned} A_{ij} &= \sum_{r=1}^N (\bar{Q}_{ij})_r (z_r - z_{r-1}), \\ B_{ij} &= \frac{1}{2} \sum_{r=1}^N (\bar{Q}_{ij})_r (z_r^2 - z_{r-1}^2), \\ D_{ij} &= \frac{1}{3} \sum_{r=1}^N (\bar{Q}_{ij})_r (z_r^3 - z_{r-1}^3). \end{aligned} \quad (6.15)$$

Here, A_{ij} are the extensional stiffness, B_{ij} the coupling stiffness, and D_{ij} the flexural stiffness.

It can be observed from Eqs. (6.13) and (6.14) that, in a general laminate, there exists a coupling between extension and flexure, that is, if an inplane force is applied, it will not only cause extensional deformation but bending and twisting as well. Further, if the layers in a laminate are laid out in a manner such that the elements of the coupling matrix B_{ij} are zero, then the laminate is termed as a *symmetric laminate*. The force and moment resultants for this laminate then reduce to

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}, \quad (6.16)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \quad (6.17)$$

On the other hand, if the layers in the laminate are such that the fibre orientation θ alternates from lamina to lamina as $+\theta/-\theta/+\theta/-\theta$, then the force and moment resultants are

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad (6.18)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \quad (6.19)$$

Such a laminate is called an *antisymmetric angle-ply laminate*. In this type of a laminate, if each lamina has the same thickness, it is then called a *regular antisymmetric angle-ply laminate*. For such a laminate, Eqs. (6.18) and (6.19) reduce to

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad (6.20)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \quad (6.21)$$

There is yet another class of laminates. Here, the laminae are oriented at either 0° or 90° . A laminate of this type is termed as a cross-ply laminate. Such a laminate can, again, be either symmetric cross-ply or antisymmetric cross-ply.

The equilibrium equations in terms of the forces (see Section 5.2) are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad (5.5)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \quad (5.6)$$

Following the same steps as in Section 5.2, and the direction of moments shown in Fig. 6.3, we can write the moment equilibrium equation as

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0, \quad (6.22)$$

where N_x , N_y , N_{xy} in Eqs. (5.5) and (5.6) are the internal forces and N_x , N_y , and N_{xy} in Eq. (6.22) are the forces applied at the edges. Substituting for N_x , N_y , N_{xy} , M_x , M_y , M_{xy}

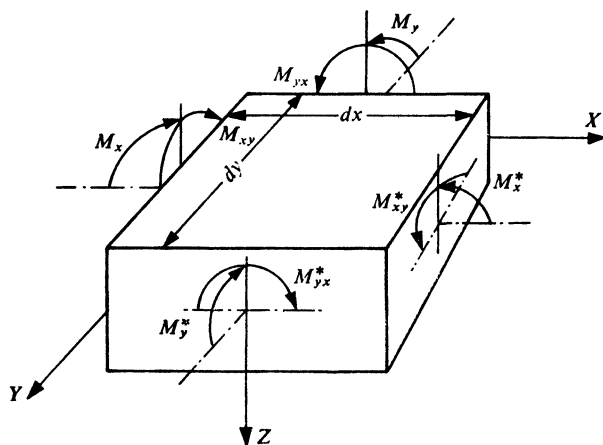


Fig. 6.3 Plate Moments (a quantity with an asterisk is as defined in Section 5.2).

from Eqs. (6.20) and (6.21), after substituting for ϵ_x^0 , ϵ_y^0 , γ_{xy}^0 , κ_x , κ_y , κ_{xy} from Eqs. (6.8) and (6.9), in Eqs. (5.5), (5.6), and (6.22), we get the governing equations as

$$\begin{aligned} & A_{11} \frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial y} + A_{16} \left(\frac{\partial^2 v^0}{\partial x^2} + 2 \frac{\partial^2 u^0}{\partial x \partial y} \right) + A_{26} \frac{\partial^2 v^0}{\partial y^2} + A_{66} \frac{\partial^2 u^0}{\partial y^2} \\ & - B_{11} \frac{\partial^3 w}{\partial x^3} - 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} - B_{26} \frac{\partial^3 w}{\partial y^3} = 0, \end{aligned} \quad (6.23a)$$

$$A_{16} \frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + A_{26} \frac{\partial^2 u^0}{\partial y^2} + A_{66} \frac{\partial^2 v^0}{\partial x^2} + 2A_{26} \frac{\partial^2 v^0}{\partial x \partial y} + A_{22} \frac{\partial^2 v^0}{\partial y^2} - B_{16} \frac{\partial^3 w}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} - B_{22} \frac{\partial^3 w}{\partial y^3} = 0, \quad (6.23b)$$

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - 3B_{16} \frac{\partial^3 u^0}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 u^0}{\partial x \partial y^2} - B_{26} \frac{\partial^3 u^0}{\partial y^3} - B_{16} \frac{\partial^3 v^0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 v^0}{\partial x \partial y^2} - B_{22} \frac{\partial^3 v^0}{\partial y^3} = -N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}. \quad (6.23c)$$

For a general laminate, all the three equations, i.e., Eqs. (6.23), have to be solved simultaneously as they are coupled. We shall subsequently consider certain special laminates, where these equations can be simplified and closed-form solutions are possible.

EXAMPLE 6.1 (Buckling of simply-supported specially orthotropic rectangular plate subjected to uniform inplane uniaxial load) Consider a laminated rectangular plate simply-supported along the edges $x = 0, a$ and $y = 0, b$ and subjected to a uniform uniaxial load N_x along the X -direction, as shown in Fig. 6.4.

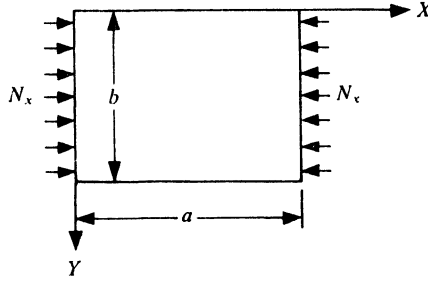


Fig. 6.4 Simply-Supported Laminated Rectangular Plate under Uniaxial Compression.

A specially orthotropic laminate is obtained when orthotropic laminae are symmetrically arranged about the middle surface of a laminate. For such a case, the stiffness coefficients A_{16} , A_{26} , D_{16} , D_{26} are zero; in addition, all the elements of the coupling matrix $[B]$ are zero. Substituting these in Eqs. (6.23), we observe that the equations governing u^0 , v^0 , and w get decoupled and can be solved independently. The buckling load is described by only the equation governing the displacement w . Therefore, we have

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} = 0. \quad (6.24)$$

This equation is identical to Eq. (5.227). In Eq. (6.24), D_{11} , D_{12} , D_{66} , and D_{22} correspond to a laminated plate, whereas, in Eq. (5.227), they correspond to a single layer of an orthotropic plate. In terms of the nondimensional parameters ξ and η , defined by Eqs. (5.228), we can write Eq. (6.24) as

$$D_{11}\frac{\partial^4 w}{\partial \xi^4} + 2p^2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + D_{22}p^4\frac{\partial^4 w}{\partial \eta^4} + N_x a^2\frac{\partial^2 w}{\partial \xi^2} = 0. \quad (6.25)$$

The simply-supported boundary conditions along the edges give

$$w = 0, \quad M_\xi = -(D_{11}\frac{\partial^2 w}{\partial \xi^2} + D_{12}p^2\frac{\partial^2 w}{\partial \eta^2}) = 0 \quad (\xi = 0, 1), \quad (6.26)$$

$$w = 0, \quad M_\eta = -(D_{12}\frac{\partial^2 w}{\partial \xi^2} + D_{22}p^2\frac{\partial^2 w}{\partial \eta^2}) = 0 \quad (\eta = 0, 1). \quad (6.27)$$

Since w is equal to zero everywhere along the edges, the moment boundary conditions reduce to

$$M_\xi = -D_{11}\frac{\partial^2 w}{\partial \xi^2} = 0, \quad (6.28)$$

$$M_\eta = -D_{22}\frac{\partial^2 w}{\partial \eta^2} = 0. \quad (6.29)$$

A solution of the type

$$w = A_{mn} \sin m\pi\xi \sin n\pi\eta \quad (6.30)$$

satisfies the boundary conditions (6.26) and (6.27) exactly. The nontrivial solution of Eq. (6.25) exists when

$$N_x = \frac{\pi^2}{a^2}[D_{11}m^2 + 2p^2(D_{11} + 2D_{66})n^2 + D_{22}p^4\frac{n^4}{m^2}]. \quad (6.31)$$

It is obvious that the smallest value of N_x is obtained when $n = 1$, so Eq. (6.31) reduces to

$$N_x = \frac{\pi^2}{a^2}[D_{11}m^2 + 2p^2(D_{11} + 2D_{66}) + D_{22}\frac{p^4}{m^2}]. \quad (6.32)$$

Thus, the minimum value of N_x for a given value of m depends on the stiffnesses and aspect ratio. When

$$\frac{D_{11}}{D_{22}} = 10, \quad \frac{D_{11} + 2D_{66}}{D_{22}} = 1,$$

Eq. (6.32) reduces to

$$N_x = \pi^2\frac{D_{22}}{a^2}[10m^2 + 2p^2 + \frac{p^4}{m^2}]. \quad (6.33)$$

The transition from m to $(m + 1)$ half waves occurs at a discrete value of p with the value of N_x remaining the same, that is,

$$10(m + 1)^2 + \frac{p^4}{(m + 1)^2} = 10m^2 + \frac{p^4}{m^2}.$$

From this equation, we obtain

$$p = (10)^{1/4} \sqrt{m(m+1)}. \quad (6.34)$$

Substituting $m = 1$, we get $p = 2.5$. At this aspect ratio, we have the transition from one half wave to two half waves. By setting $m = 2$, we can obtain the transition from two to three half waves and this occurs when $p = 4.35$.

The buckling load for a square plate ($p = 1$) is obtained by putting $m = 1$:

$$(N_x)_{cr} = \frac{13\pi^2 D_{22}}{b^2} = k \frac{\pi^2 D_{22}}{b^2}. \quad (6.35)$$

Table 6.1 gives the values of k for different values of aspect ratio.

Table 6.1 Variation of k with Aspect Ratio p

m	1	1	1	2	2	3
p	0.5	1	2	3	4	5
k	42.25	13.00	8.50	8.96	8.50	8.37

EXAMPLE 6.2 (Buckling of simply-supported antisymmetric angle-ply laminate subjected to uniform inplane uniaxial load) Consider the laminated plate shown in Fig. 6.4. An antisymmetric angle-ply laminate has the nonzero stiffnesses A_{11} , A_{12} , A_{22} , A_{66} , B_{16} , B_{26} , D_{11} , D_{22} , and D_{66} as given in Eqs. (6.20) and (6.21). Further, the governing equations are

$$A_{11} \frac{\partial^2 u^0}{\partial x^2} + A_{66} \frac{\partial^2 u^0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial y} - 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + B_{26} \frac{\partial^3 w}{\partial y^3} = 0, \quad (6.36)$$

$$(A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + A_{66} \frac{\partial^2 v^0}{\partial x^2} + A_{22} \frac{\partial^2 v^0}{\partial y^2} - B_{16} \frac{\partial^3 w}{\partial x^3} - 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} = 0, \quad (6.37)$$

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{16} (3 \frac{\partial^3 u^0}{\partial x^2 \partial y} + \frac{\partial^3 v^0}{\partial x^3}) - B_{26} (\frac{\partial^3 u^0}{\partial y^3} + 3 \frac{\partial^3 v^0}{\partial x \partial y^2}) + N_x \frac{\partial^2 w}{\partial x^2} = 0. \quad (6.38)$$

Thus, the equations for u^0 , v^0 , and w are coupled. In such a situation, it is not easy to find the solution for u^0 , v^0 , and w since the boundary conditions on each of these displacements have to be satisfied. We shall therefore so choose the boundary conditions that a closed-form solution is possible. (For boundary conditions other than those we select, it is difficult, if not impossible, to apply this technique.) These boundary conditions are

$$w = 0, \quad M_x = B_{16} \left(\frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y} \right) - D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} = 0 \quad (6.39a)$$

$$(x = 0, a),$$

$$u^0 = 0, \quad N_{xy} = A_{66} \left(\frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y} \right) - B_{16} \frac{\partial^2 w}{\partial x^2} - B_{26} \frac{\partial^2 w}{\partial y^2} = 0 \quad (6.39b)$$

$$w = 0, \quad M_y = B_{26}\left(\frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} = 0 \quad (6.39c)$$

(y = 0, b).

$$v^0 = 0, \quad N_{xy} = A_{66}\left(\frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y}\right) - B_{16}\frac{\partial^2 w}{\partial x^2} - B_{26}\frac{\partial^2 w}{\partial y^2} = 0 \quad (6.39d)$$

Introducing the nondimensional parameters, ξ and η , we can rewrite the governing equations and the boundary conditions as

$$A_{11}\frac{\partial^2 u^0}{\partial \xi^2} + A_{66}p^2\frac{\partial^2 u^0}{\partial \eta^2} + p(A_{12} + A_{66})\frac{\partial^2 v^0}{\partial \xi \partial \eta} - \frac{3}{b}B_{16}\frac{\partial^3 w}{\partial \xi^2 \partial \eta} - B_{26}\frac{p^2}{b}\frac{\partial^3 w}{\partial \eta^3} = 0, \quad (6.40)$$

$$p(A_{12} + A_{66})\frac{\partial^2 u^0}{\partial \xi \partial \eta} + A_{66}\frac{\partial^2 v^0}{\partial \xi^2} + A_{22}p^2\frac{\partial^2 v^0}{\partial \eta^2} - \frac{B_{16}}{a}\frac{\partial^3 w}{\partial \xi^3} - 3B_{26}\frac{p}{b}\frac{\partial^3 w}{\partial \xi \partial \eta^2} = 0, \quad (6.41)$$

$$D_{11}\frac{\partial^4 w}{\partial \xi^4} + 2(D_{12} + 2D_{66})p^2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + D_{22}p^4\frac{\partial^4 w}{\partial \eta^4} - B_{16}(3ap\frac{\partial^3 u^0}{\partial \xi^2 \partial \eta} + a\frac{\partial^3 v^0}{\partial \xi^3}) \\ - B_{26}(p^3a\frac{\partial^3 u^0}{\partial \eta^3} + 3p^2a\frac{\partial^3 v^0}{\partial \xi \partial \eta^2}) + N_x a^2\frac{\partial^2 w}{\partial \xi^2} = 0, \quad (6.42)$$

$$w = 0, \quad M_\xi = B_{16}\left(a\frac{\partial v^0}{\partial \xi} + ap\frac{\partial u^0}{\partial \eta}\right) - D_{11}\frac{\partial^2 w}{\partial \xi^2} - D_{12}p^2\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (6.43a)$$

(\xi = 0, 1),

$$u^0 = 0, \quad N_{\xi\eta} = A_{66}\left(a\frac{\partial v^0}{\partial \xi} + ap\frac{\partial u^0}{\partial \eta}\right) - B_{16}\frac{\partial^2 w}{\partial \xi^2} - B_{26}p^2\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (6.43b)$$

$$w = 0, \quad M_\eta = B_{26}\left(a\frac{\partial v^0}{\partial \xi} + ap\frac{\partial u^0}{\partial \eta}\right) - D_{12}\frac{\partial^2 w}{\partial \xi^2} - D_{22}p^2\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (6.43c)$$

(\eta = 0, 1).

$$v^0 = 0, \quad N_{\xi\eta} = A_{66}\left(a\frac{\partial v^0}{\partial \xi} + ap\frac{\partial u^0}{\partial \eta}\right) - B_{16}\frac{\partial^2 w}{\partial \xi^2} - B_{26}p^2\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (6.43d)$$

Our method of finding the solution here is similar to that in Jones et al [2]. If u^0 , v^0 , and w are so chosen that

$$\begin{aligned} u^0 &= \bar{u} \sin m\pi\xi \cos n\pi\eta, \\ v^0 &= \bar{v} \cos m\pi\xi \sin n\pi\eta, \\ w &= \bar{w} \sin m\pi\xi \sin n\pi\eta, \end{aligned} \quad (6.44)$$

then the boundary conditions (6.43) are satisfied exactly. Substituting Eqs. (6.44) in the differential equations (6.40)–(6.42), we get

$$\bar{u}(A_{11}m^2\pi^2 + A_{66}p^2n^2\pi^2) + \bar{v}(A_{12} + A_{66})pmn\pi^2 - \bar{w}(3B_{16}m^2\pi^2 + B_{26}p^2n^2\pi^2)\frac{n\pi}{b} = 0, \quad (6.45a)$$

$$\bar{u}(A_{12} + A_{66})mn\pi^2 + \bar{v}(A_{22}p^2n^2\pi^2 + A_{66}m^2\pi^2) - \bar{w}(B_{16}m^2\pi^2 + 3B_{26}p^2n^2\pi^2)\frac{m\pi}{a} = 0, \quad (6.45b)$$

$$\begin{aligned}
& -\bar{u}(3B_{16}m^2\pi^2 + B_{26}pn^2\pi^2)\frac{n\pi}{b}a^2 - \bar{v}(B_{16}m^2\pi^2 + 3B_{26}p^2n^2\pi^2)\frac{m\pi}{a}a^2 \\
& + \bar{w}[D_{11}m^4\pi^4 + 2p^2m^2n^2\pi^2(D_{12} + 2D_{66}) + D_{22}p^4n^4\pi^4] - N_x a^2 m^2 n^2 \bar{w} = 0. \quad (6.45c)
\end{aligned}$$

Equations (6.45) are a set of simultaneous algebraic equations; hence, for a nontrivial solution, the determinant of the coefficients of $\bar{u}$, $\bar{v}$, $\bar{w}$ must be zero. This leads to the characteristic equation from which $(N_x)_{cr}$ can be obtained as

$$(N_x)_{cr} = \frac{1}{a^2 m^2 n^2} [T_{33} + \frac{a^2(2T_{12}T_{13}T_{23} - T_{11}T_{23}^2 - T_{13}^2T_{22})}{T_{11}T_{22} - T_{12}^2}], \quad (6.46)$$

where

$$\begin{aligned}
T_{11} &= A_{11}m^2\pi^2 + A_{66}p^2n^2\pi^2, \\
T_{12} &= (A_{12} + A_{66})pmn\pi^2, \\
T_{13} &= -(3B_{16}m^2\pi^2 + B_{26}p^2n^2\pi^2)(\frac{n\pi}{b}), \\
T_{22} &= (A_{22}p^2n^2\pi^2 + A_{66}m^2\pi^2), \\
T_{23} &= -(B_{16}m^2\pi^2 + 3B_{26}p^2n^2\pi^2)(\frac{m\pi}{a}), \\
T_{33} &= D_{11}m^4\pi^4 + 2p^2m^2n^2\pi^2(D_{12} + 2D_{66}) + D_{22}p^4n^4\pi^4.
\end{aligned} \quad (6.47)$$

As can be seen from Eq. (6.46), the buckling load is a function of A_{ij} , B_{ij} , and D_{ij} and the aspect ratio of the plate. These, in turn, depend on the number of layers, and the thickness and mechanical properties of each layer in the laminate. In our analysis, we have assumed that, for each layer, the material and the thickness are the same.

Table 6.2 shows the variation of critical load with the orientation and number of layers. Clearly, as the number of layers increases, the buckling load also increases. However, the difference in the buckling load for two layers and four layers is initially large, but decreases

Table 6.2 Variation of Buckling Load $[N_x/(E_T t^3)]$ with Orientation and Number of Layers (aspect ratio 1.0, $E_L/E_T = 6.3334$, $G_{LT}/E_T = 0.6667$, $\nu_{LT} = 0.2$, $\nu_{TL} = 0.0315$)

Number of layers	$\theta = 0^\circ$	$\theta = 30^\circ$	$\theta = 45^\circ$	$\theta = 60^\circ$	$\theta = 90^\circ$
2	0.6872	0.6213	0.5857	0.6213	0.6872
4	0.6873	0.8408	0.8743	0.8408	0.6873
6	0.6874	0.8654	0.9283	0.8654	0.6874
8	0.6875	0.8809	0.9470	0.8809	0.6875

as the number of layers increases. Moreover, the buckling load is maximum when $\theta = 45^\circ$ except when the number of layers is two.

6.3 HIGHER ORDER SINGLE GOVERNING EQUATION

The method of analysis we have given is useful if we can find out a suitable function separately for each of the three displacement components u^0 , v^0 , and w . However, to obtain a more general solution for complicated boundary conditions, it is advisable that the three displacement equations of equilibrium be combined into a single, higher order partial differential equation. In Sharma et al [3, 4, 5], the technique of doing this has been discussed in detail and applied to different sets of boundary conditions. In what follows, we shall reduce the three governing equations, namely, (6.23), to a single, eighth order partial differential equation in terms of a single displacement function. Equations (6.23a), (6.23b), and (6.23c) can be rewritten as

$$L_{11}u^0 + L_{12}v^0 + L_{13}w = 0, \quad (6.48a)$$

$$L_{21}u^0 + L_{22}v^0 + L_{23}w = 0, \quad (6.48b)$$

$$L_{31}u^0 + L_{32}v^0 + L_{33}w = -N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}, \quad (6.48c)$$

where

$$L_{11} = A_{11} \frac{\partial^2}{\partial x^2} + 2A_{16} \frac{\partial^2}{\partial x \partial y} + A_{66} \frac{\partial^2}{\partial y^2}, \quad (6.49a)$$

$$L_{21} = L_{12} = A_{16} \frac{\partial^2}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2}{\partial x \partial y} + A_{26} \frac{\partial^2}{\partial y^2}, \quad (6.49b)$$

$$L_{31} = L_{13} = -B_{11} \frac{\partial^3}{\partial x^3} - 3B_{16} \frac{\partial^3}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3}{\partial x \partial y^2} - B_{26} \frac{\partial^3}{\partial y^3}, \quad (6.49c)$$

$$L_{22} = A_{66} \frac{\partial^2}{\partial x^2} + 2A_{26} \frac{\partial^2}{\partial x \partial y} + A_{22} \frac{\partial^2}{\partial y^2}, \quad (6.49d)$$

$$L_{32} = L_{23} = -B_{16} \frac{\partial^3}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3}{\partial x^2 \partial y} - 3B_{26} \frac{\partial^3}{\partial x \partial y^2} - B_{22} \frac{\partial^3}{\partial y^3}, \quad (6.49e)$$

$$L_{33} = D_{11} \frac{\partial^4}{\partial x^4} + 4D_{16} \frac{\partial^4}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4}{\partial x \partial y^3} + D_{22} \frac{\partial^4}{\partial y^4}. \quad (6.49f)$$

Let us assume

$$u^0 = -L_{12}\eta(x, y) - L_{13}\psi(x, y), \quad (6.50a)$$

$$v^0 = L_{11}\eta(x, y), \quad (6.50b)$$

$$w = L_{11}\psi(x, y), \quad (6.50c)$$

where the functions $\eta(x, y)$ and $\psi(x, y)$ are continuous and differentiable in the domain of the laminate. These displacements satisfy Eq. (6.48a) identically, whereas Eq. (6.48b) becomes

$$(L_{22}L_{11} - L_{12}L_{21})\eta + (L_{23}L_{11} - L_{13}L_{21})\psi = 0. \quad (6.51)$$

Equation (6.51) can be rewritten as

$$P\eta + Q\psi = 0. \quad (6.52)$$

If we now choose

$$\eta = -Q\Phi(x, y), \quad (6.53a)$$

$$\psi = P\Phi(x, y), \quad (6.53b)$$

Eq. (6.48b) is identically satisfied. This enables us to express Eqs. (6.50a), (6.50b), and (6.50c), in terms of ϕ alone, as

$$u^0 = (L_{12}L_{23} - L_{13}L_{22})\phi(x, y), \quad (6.54a)$$

$$v^0 = (L_{13}L_{21} - L_{23}L_{11})\phi(x, y), \quad (6.54b)$$

$$w = (L_{11}L_{22} - L_{12}L_{21})\phi(x, y), \quad (6.54c)$$

where

$$\phi(x, y) = L_{11}\Phi(x, y).$$

Substituting Eqs. (6.54) in Eq. (6.48c), we have

$$\begin{aligned} & (L_{11}L_{22}L_{33} - L_{21}L_{12}L_{33} - L_{32}L_{23}L_{11} + L_{31}L_{12}L_{23} + L_{32}L_{13}L_{21} - L_{31}L_{13}L_{22})\phi(x, y) \\ & = -(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy} \frac{\partial^2}{\partial x \partial y} + N_y \frac{\partial^2}{\partial y^2})(L_{11}L_{22} - L_{12}L_{21})\phi. \end{aligned} \quad (6.55)$$

Equation (6.55), when expanded, becomes

$$\begin{aligned} & A_{11} \frac{\partial^8 \phi}{\partial x^8} + A_2 \frac{\partial^8 \phi}{\partial x^7 \partial y} + A_3 \frac{\partial^8 \phi}{\partial x^6 \partial y^2} + A_4 \frac{\partial^8 \phi}{\partial x^5 \partial y^3} + A_5 \frac{\partial^8 \phi}{\partial x^4 \partial y^4} \\ & + A_6 \frac{\partial^8 \phi}{\partial x^3 \partial y^5} + A_7 \frac{\partial^8 \phi}{\partial x^2 \partial y^6} + A_8 \frac{\partial^8 \phi}{\partial x \partial y^7} + A_9 \frac{\partial^8 \phi}{\partial y^8} \\ & = -(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy} \frac{\partial^2}{\partial x \partial y} + N_y \frac{\partial^2}{\partial y^2}) \{ (A_{11}A_{66} - A_{16}^2) \frac{\partial^4 \phi}{\partial x^4} \\ & + 2(A_{11}A_{26} - A_{12}A_{16}) \frac{\partial^4 \phi}{\partial x^3 \partial y} + (A_{11}A_{22} + 2A_{16}A_{26} - 2A_{12}A_{66} - A_{12}^2) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \\ & + 2(A_{16}A_{22} - A_{12}A_{26}) \frac{\partial^4 \phi}{\partial x \partial y^3} + (A_{66}A_{22} - A_{26}^2) \frac{\partial^4 \phi}{\partial y^4} \}, \end{aligned} \quad (6.56)$$

where

$$A_1 = B_{11}(2A_{16}B_{16} - A_{66}B_{11}) - A_{11}B_{16}^2 + D_{11}(A_{11}A_{66} + A_{16}^2), \quad (6.57a)$$

$$\begin{aligned} A_2 = & 2B_{11}(A_{16}G + E_1B_{16} - 3A_{66}B_{16} - A_{26}B_{11}) + 2B_{16}(2A_{16}B_{16} - A_{11}G) \\ & + 2D_{11}(A_{16}A_{66} + A_{11}A_{26} - A_{16}E_1) + 4D_{16}(A_{11}A_{66} - A_{16}^2), \end{aligned} \quad (6.57b)$$

$$\begin{aligned}
A_3 = & 2B_{11}(3A_{16}B_{26} + E_1G - 5A_{26}B_{16} - A_{66}G) + 2B_{16}(2A_{16}G + 3E_1B_{16} \\
& - 5A_{66}B_{16} - 3A_{11}B_{26}) + D_{11}(2A_{16}A_{26} + A_{66}^2 + A_{11}A_{22} - E_1^2) \\
& + 8D_{16}(A_{16}A_{66} + A_{11}A_{26} - A_{16}E_1) - A_{22}B_{11}^2 \\
& - A_{11}G^2 + 2F(A_{11}A_{66} - A_{16}^2), \tag{6.57c}
\end{aligned}$$

$$\begin{aligned}
A_4 = & 2B_{11}(A_{16}B_{22} + 3E_1B_{26} - A_{66}B_{66} - 3A_{22}B_{16} - A_{26}G) - 2B_{22}B_{16}A_{11} \\
& + 4B_{16}(2A_{16}B_{26} - 2E_1G - 3A_{26}B_{16} - 2A_{66}G) - 6B_{26}A_{11}G \\
& + 2D_{11}(A_{22}A_{16} + A_{26}A_{66} - A_{26}E_1) + 4D_{16}(A_{66}^2 + A_{11}A_{22} + 2A_{16}A_{26} - E_1^2) \\
& + 4D_{26}(A_{11}A_{66} - A_{16}^2) + 4F(A_{16}A_{66} + A_{11}A_{26} - A_{16}E_1), \tag{6.57d}
\end{aligned}$$

$$\begin{aligned}
A_5 = & 2B_{11}(E_1B_{22} + A_{26}B_{26} - A_{22}G) + 2B_{22}(A_{16}B_{16} - A_{11}G) - 4B_{16}A_{26}G \\
& - 4B_{26}A_{16}G - 9A_{22}B_{16}^2 - 9A_{11}B_{26}^2 + 4B_{16}B_{26}(5E_1 - 3A_{66}) + D_{11}(A_{22}A_{66} - A_{26}^2) \\
& + D_{22}(A_{11}A_{66} - A_{16}^2) + 8D_{16}(A_{22}A_{16} + A_{26}A_{66} - A_{26}E_1) \\
& + 8D_{26}(A_{16}A_{66} + A_{11}A_{26} - A_{16}E_1) + 2G(E_1^2 - A_{66}G) \\
& + 2F(A_{66}^2 + A_{11}A_{22} + 2A_{16}A_{26} - E_1^2), \tag{6.57e}
\end{aligned}$$

$$\begin{aligned}
A_6 = & 2B_{11}(A_{26}B_{22} - A_{22}B_{26}) + 2B_{22}(2E_1B_{16} - A_{16}G - 3A_{11}B_{26} - A_{66}B_{16}) \\
& + 2B_{16}(4A_{26}B_{26} - 3A_{22}G) + 4B_{26}(2E_1G - 2A_{66}G - 3A_{16}B_{26}) + 2D_{22}(A_{16}A_{66} \\
& + A_{11}A_{26} - A_{16}E_1) + 4D_{16}(A_{22}A_{66} - A_{26}^2) + 4D_{26}(A_{11}A_{22} + 2A_{16}A_{26} \\
& + 2A_{66}^2 - 2E_1^2) + 4F(A_{22}A_{16} + A_{22}A_{66} - A_{26}E_1), \tag{6.57f}
\end{aligned}$$

$$\begin{aligned}
A_7 = & B_{22}(6A_{26}B_{16} + 2E_1G - 10A_{16}B_{26} - 2A_{66}G - A_{11}B_{22}) - 6B_{16}B_{26}A_{22} \\
& + 2B_{26}^2(3E_1 - 5A_{66}) + D_{22}(A_{11}A_{22} + A_{66}^2 - E_1^2 + 2A_{16}A_{26}) + 8D_{26}(A_{22}A_{16} \\
& + A_{26}A_{66} - A_{26}E_1) + 2F(A_{22}A_{66} - A_{26}^2) + G(4A_{26}B_{26} - A_{22}G), \tag{6.57g}
\end{aligned}$$

$$\begin{aligned}
A_8 = & 2B_{22}(A_{26}G + B_{26}E_1 - 3A_{66}B_{26} - A_{16}B_{22}) + 2B_{26}(2B_{26}A_{26} - GA_{22}) \\
& + 2D_{22}(A_{22}A_{16} + A_{26}A_{66} - A_{26}E_1) + 4D_{26}(A_{66}A_{22} - A_{26}^2), \tag{6.57h}
\end{aligned}$$

$$A_9 = B_{22}(2A_{26}B_{26} - B_{22}A_{66}) - A_{22}B_{26}^2 + D_{22}(A_{22}A_{66} - A_{26}^2), \tag{6.57i}$$

$$E_1 = A_{12} + A_{66}, \quad F = D_{12} + 2D_{66}, \quad G = B_{12} + 2B_{66}. \tag{6.57j}$$

Equation (6.56) is the governing equation for buckling of laminated plates.

For an asymmetric angle-ply plate, the following relations hold [see Eqs. (6.20) and (6.21)]:

$$A_{16} = A_{26} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0. \tag{6.58}$$

In view of Eqs. (6.58), the coefficients A_2 , A_4 , A_6 , and A_8 vanish in Eq. (6.56). Thus, the governing equation for buckling of an antisymmetric angle-ply laminate is

$$\begin{aligned} & A_1 \frac{\partial^8 \phi}{\partial x^8} + A_3 \frac{\partial^8 \phi}{\partial x^6 \partial y^2} + A_5 \frac{\partial^8 \phi}{\partial x^4 \partial y^4} + A_7 \frac{\partial^8 \phi}{\partial x^2 \partial y^6} + A_9 \frac{\partial^8 \phi}{\partial y^8} \\ &= -(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy} \frac{\partial^2}{\partial x \partial y} + N_y \frac{\partial^2}{\partial y^2}) \{ (A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \\ &+ A_{66}A_{22} \frac{\partial^4 \phi}{\partial y^4} + A_{11}A_{66} \frac{\partial^4 \phi}{\partial x^4} \}, \end{aligned} \quad (6.59)$$

where A_1 , A_3 , A_5 , A_7 , and A_9 are obtained after substituting Eqs. (6.58) in Eq. (6.56).

For an antisymmetric cross-ply plate, the following relations hold (Jones [1]):

$$A_{16} = A_{26} = D_{16} = D_{26} = B_{12} = B_{16} = B_{26} = B_{66} = 0, \quad B_{22} = -B_{11}. \quad (6.60)$$

In view of relations (6.60), the coefficients A_2 , A_4 , A_6 , and A_8 vanish in Eq. (6.56). The governing equation for buckling of an antisymmetric cross-ply then is the same as Eq. (6.59). However, the coefficients A_1 to A_9 are not the same. These can be obtained by substituting Eqs. (6.60) in Eqs. (6.57).

We shall illustrate the use of the higher order governing equation (6.56) through examples.

EXAMPLE 6.3 (Buckling of laminated angle-ply composite plate with loading edges simply-supported and other two edges free) Consider a laminated rectangular plate simply-supported along the edges $x = 0$, a and subjected to a uniform uniaxial load N_x along the

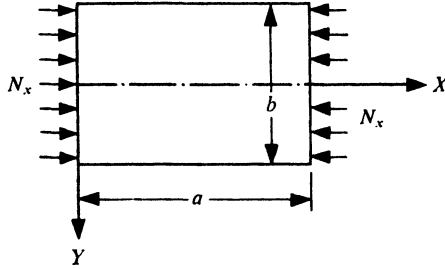


Fig. 6.5 Laminated Plate under Uniaxial Compression.

X -direction, as shown in Fig. 6.5. For such a plate, the governing equation (6.59) reduces to

$$\begin{aligned} & A_1 \frac{\partial^8 \phi}{\partial x^8} + A_3 \frac{\partial^8 \phi}{\partial x^6 \partial y^2} + A_5 \frac{\partial^8 \phi}{\partial x^4 \partial y^4} + A_7 \frac{\partial^8 \phi}{\partial x^2 \partial y^6} + A_9 \frac{\partial^8 \phi}{\partial y^8} + N_x \{ A_{16}A_{66} \frac{\partial^6 \phi}{\partial x^6} \\ &+ (A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2) \frac{\partial^6 \phi}{\partial x^4 \partial y^2} + A_{22}A_{66} \frac{\partial^6 \phi}{\partial x^2 \partial y^4} \} = 0. \end{aligned} \quad (6.61)$$

In terms of the nondimensional parameters, defined by Eqs. (5.228), Eq. (6.61) can be written as

$$\begin{aligned}
& A_1 \frac{\partial^8 \phi}{\partial \xi^8} + A_3 p^2 \frac{\partial^8 \phi}{\partial \xi^6 \partial \eta^2} + A_5 p^4 \frac{\partial^8 \phi}{\partial \xi^4 \partial \eta^4} + A_7 p^6 \frac{\partial^8 \phi}{\partial \xi^2 \partial \eta^6} + A_9 p^8 \frac{\partial^8 \phi}{\partial \eta^8} \\
& + N_x a^2 \{ A_{11} A_{66} \frac{\partial^6 \phi}{\partial \xi^6} + p^2 (A_{11} A_{22} - 2 A_{12} A_{66} - A_{12}^2) \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + A_{22} A_{66} p^4 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} \} = 0.
\end{aligned} \quad (6.62)$$

The boundary conditions can also be written in the nondimensional form. Thus, for $\xi = 0, 1$, we have

$$u^0 = B_{16}(2A_{66} - A_{12}) \frac{\partial^5 \phi}{\partial \xi^3 \partial \eta^2} - B_{26}(2A_{66} + 3A_{12} - 3A_{11}) p^2 \frac{\partial^5 \phi}{\partial \xi^2 \partial \eta^3} + B_{22} A_{22} p^4 \frac{\partial^5 \phi}{\partial \eta^5} = 0, \quad (6.63a)$$

$$w = A_{11} A_{66} \frac{\partial^4 \phi}{\partial \xi^4} + A_{66} A_{22} p^4 \frac{\partial^4 \phi}{\partial \eta^4} + (A_{11} A_{22} - 2 A_{12} A_{66} - A_{12}^2) p^2 \frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} = 0, \quad (6.63b)$$

$$M_\xi = a_{11} \frac{\partial^6 \phi}{\partial \xi^6} + a_{12} p^2 \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{13} p^4 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} + a_{14} p^6 \frac{\partial^6 \phi}{\partial \eta^6} = 0, \quad (6.63c)$$

$$N_{\xi\eta} = a_{15} \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{16} p^2 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} = 0 \quad (6.63d)$$

and for $\eta = \pm 1/2$, we have

$$N_\eta = a_{17} \frac{\partial^6 \phi}{\partial \xi^5 \partial \eta} + a_{18} p^2 \frac{\partial^6 \phi}{\partial \xi^3 \partial \eta^3} = 0, \quad (6.63e)$$

$$N_{\xi\eta} = a_{17} \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{16} p^2 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} = 0, \quad (6.63f)$$

$$M_\eta = a_{19} \frac{\partial^6 \phi}{\partial \xi^6} + a_{20} p^2 \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{21} p^4 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} + a_{22} p^6 \frac{\partial^6 \phi}{\partial \eta^6} = 0, \quad (6.63g)$$

$$2 \frac{\partial M_{\xi\eta}}{\partial \xi} + p \frac{\partial M_\eta}{\partial \eta} = a_{24} \frac{\partial^7 \phi}{\partial \xi^6 \partial \eta} + a_{25} p^2 \frac{\partial^7 \phi}{\partial \xi^4 \partial \eta^3} + a_{26} p^4 \frac{\partial^7 \phi}{\partial \xi^2 \partial \eta^5} + a_{23} p^6 \frac{\partial^7 \phi}{\partial \eta^7} = 0, \quad (6.63h)$$

where

$$a_{11} = A_{11}(B_{16}^2 - A_{66}D_{11}), \quad (6.64a)$$

$$a_{12} = 3A_{11}B_{16}B_{26} - 4B_{16}^2A_{12} - D_{11}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - D_{12}A_{11}A_{66}, \quad (6.64b)$$

$$a_{13} = B_{16}(3B_{16}A_{22} - 4B_{26}A_{12}) - D_{12}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - D_{11}A_{22}A_{66}, \quad (6.64c)$$

$$a_{14} = A_{22}(B_{16}B_{26} - D_{12}A_{66}), \quad (6.64d)$$

$$a_{15} = 2B_{26}A_{11}A_{66} \underset{\sim}{=} B_{16}(A_{12}^2 - 2A_{12}A_{66} - A_{11}A_{22}), \quad (6.64e)$$

$$a_{16} = 2B_{16}A_{22}A_{66} + B_{26}(A_{12}^2 - 2A_{12}A_{66} - A_{11}A_{22}), \quad (6.64f)$$

$$a_{17} = -a_{15}, \quad (6.64g)$$

$$a_{18} = -a_{16}, \quad (6.64h)$$

$$a_{19} = A_{11}(B_{16}B_{26} - A_{66}D_{12}), \quad (6.64i)$$

$$a_{20} = A_{11}(3B_{26}^2 - A_{66}A_{22}) - 4B_{16}B_{26}A_{12} - D_{12}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}), \quad (6.64j)$$

$$a_{21} = A_{22}(3B_{16}B_{26} - D_{12}A_{66}) - 4B_{26}^2A_{12} - D_{22}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}), \quad (6.64k)$$

$$a_{22} = A_{22}(B_{26}^2 - D_{22}A_{66}), \quad (6.64l)$$

$$a_{23} = a_{22}, \quad (6.64m)$$

$$a_{24} = 3A_{11}B_{16}B_{26} - A_{11}A_{66}D_{12} - 4A_{11}D_{66}A_{66} + 4B_{12}^2A_{66} - 2A_{12}B_{16}^2, \quad (6.64n)$$

$$a_{25} = 9B_{26}^2A_{11} - 16B_{16}B_{26}A_{12} - (D_{12} + 4D_{66})(A_{11}A_{22}A_{12}^2 - 2A_{12}A_{66}) \\ + 6B_{16}^2A_{22} + 8B_{16}B_{26}A_{66}, \quad (6.64o)$$

$$a_{26} = A_{22}(5B_{16}B_{26} - D_{12}A_{66}) - 6B_{26}^2A_{12} + 4B_{26}^2A_{66} - D_{22}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) \\ - 4D_{66}A_{22}A_{66}. \quad (6.64p)$$

The solution for $\phi(\xi, \eta)$ is so chosen that it satisfies the boundary conditions at $\xi = 0, 1$ identically. The displacement function can therefore be written as

$$\phi(\xi, \eta) = \phi_m(\eta) \sin(\alpha_m \xi), \quad (6.65)$$

where $\alpha_m = m\pi$. This method of analysis is generally referred to as Levy's method. When expression (6.65) is substituted in Eq. (6.62), the resulting equation for $\phi_m(\eta)$ is an ordinary differential equation and has a solution of the type

$$\phi_m(\eta) = e^{\lambda \alpha_m \eta}. \quad (6.66)$$

The solution of Eq. (6.62) can therefore be written in the form

$$\phi_m(\xi, \eta) = e^{\lambda \alpha_m \eta} \sin(\alpha_m \xi). \quad (6.67)$$

Substituting relation (6.67) in Eq. (6.62), we get the algebraic equation

$$\lambda^8 + c_1\lambda^6 + c_2\lambda^4 + c_3\lambda^2 + c_4 = 0, \quad (6.68)$$

where

$$c_1 = -A_7/A_9, \quad c_2 = (A_5 - \mu p^4 A_{22}A_{66})/A_9, \\ c_3 = [-A_3 + \mu p^2(A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2)]/A_9, \quad c_4 = (A_1 - \mu A_{16}A_{66})/A_9, \\ \mu = N_x a^2 / \alpha_m^2.$$

Equation (6.68) gives eight roots of λ which can be real or complex. As the laminate is symmetric about the X -axis, we need consider only one-half of the laminate. Applying the boundary conditions along the edge $\eta = 1/2$, we get four homogeneous algebraic equations in terms of $c_1, \dots, c_4$ whose coefficients are the functions of N_x , laminate material properties, and number of layers in the laminate. Since the equations so obtained are complex, the critical load N_x has to be calculated using a computer.

We now discuss the results of the numerical computations we carried out for different values of the aspect ratio of an angle-ply laminate of carbon fibre reinforced plastic (CFRP) with varying number of laminae. The material properties are $E_1/E_2 = 6.33$, $G_{12}/E_2 = 0.667$, and $\nu_{12} = 0.20$.

Figure 6.6 shows the nondimensional buckling load as a function of the thickness ratio t_1/t_2 of a four-layered antisymmetric angle-ply square plate for three different fibre orientations, namely, 30° , 45° , and 60° . As can be seen, for each orientation, the buckling load is maximum at $t_1/t_2 = 0.4142$. This is due to the fact that the coupling stiffnesses (B_{16} and B_{26}) tend to vanish at this thickness ratio.

Figures 6.7 and 6.8 show the results for a six-layered antisymmetric angle-ply square plate. Here, the buckling load is maximum at $t_2/t_3 = 2.0$ for $t_1/t_3 = 1.0$ (Fig. 6.7), and $t_2/t_3 = 4.1623$ for $t_1/t_3 = 2.0$ (Fig. 6.8).

Figures 6.9, 6.10, and 6.11 exhibit the variation of buckling load as a function of the aspect ratio $p (=a/b)$ for an antisymmetric angle-ply plate for three different fibre orientations. Clearly, as the number of layers increases, the buckling load tends to be independent of the number of layers.

Coupling brings down the strength of a laminate. This effect of coupling can be reduced or eliminated by a proper choice of stacking sequence or by increasing the number of layers (Sharma et al [3]). The effect of coupling as a percentage reduction in the buckling load between four- and two-layered, and between six- and four-layered configurations of a square antisymmetric angle-ply plate as a function of θ is given in Table 6.3. As can be seen from this table, the reduction in the buckling load is smaller at extreme values of θ (close to 0° and 90°). This is because at these values the coupling terms tend to vanish.

Table 6.3 Percentage Reduction in Buckling Load As A Consequence of Coupling for CFRP Square Antisymmetric Angle-Ply Plate

Fibre orientation (θ)	Percentage reduction between $N = 4$ and $N = 2$	Percentage reduction between $N = 6$ and $N = 4$
15	6.73	2.32
20	18.93	3.24
25	22.39	3.90
30	23.92	4.32
35	23.97	4.20
40	22.39	3.93
45	19.67	3.34
50	16.18	2.89
55	12.27	2.23
60	8.87	1.63
65	5.36	1.14
70	3.63	0.98
75	1.29	0.40

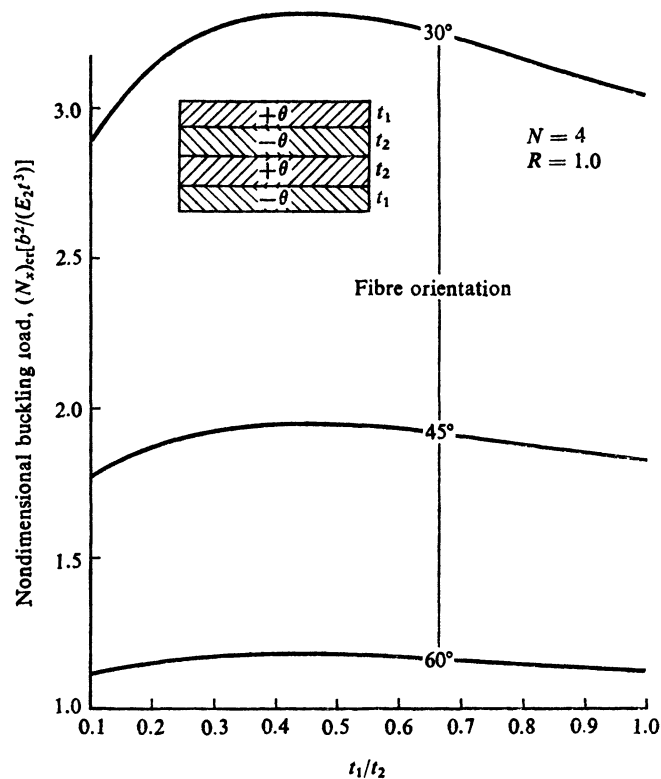


Fig. 6.6 Variation of Buckling Load with Thickness Ratio.

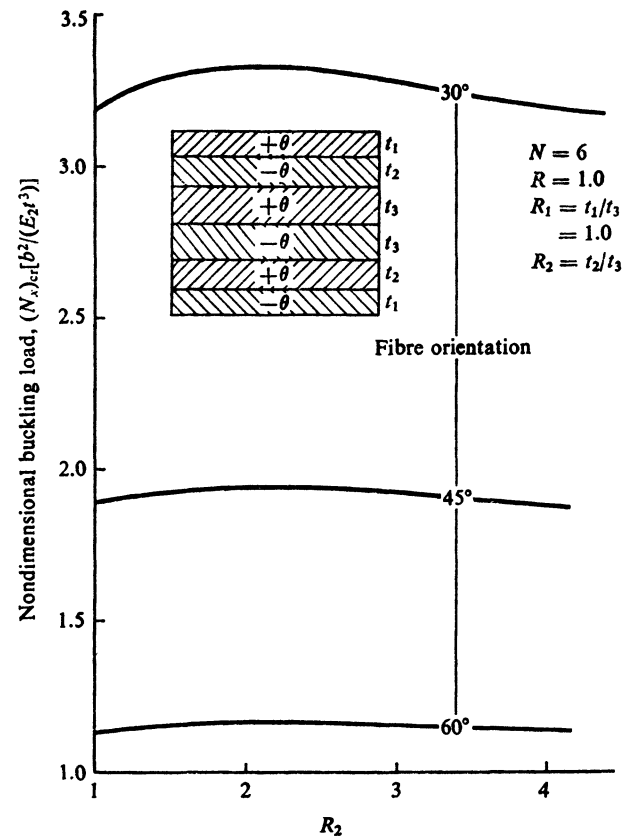


Fig. 6.7 Variation of Buckling Load with Thickness Ratio.

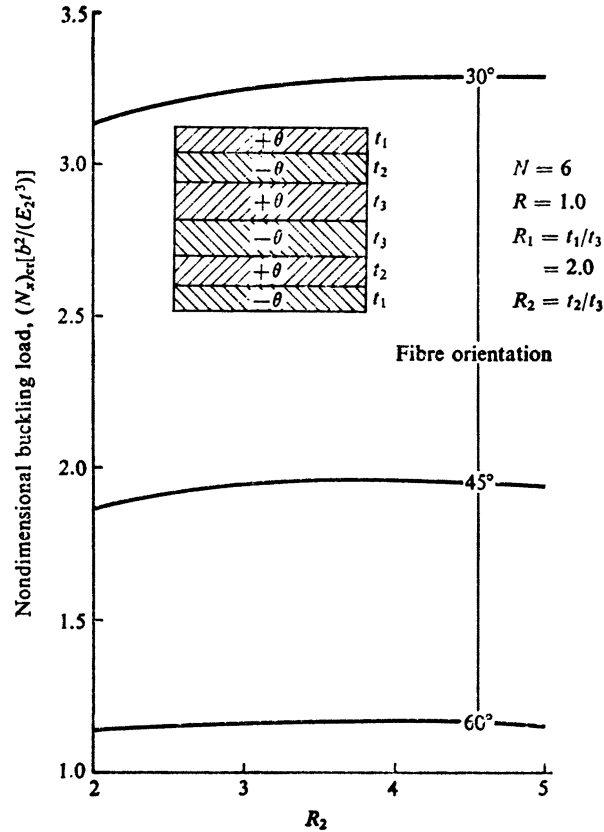


Fig. 6.8 Variation of Buckling Load with Thickness Ratio.

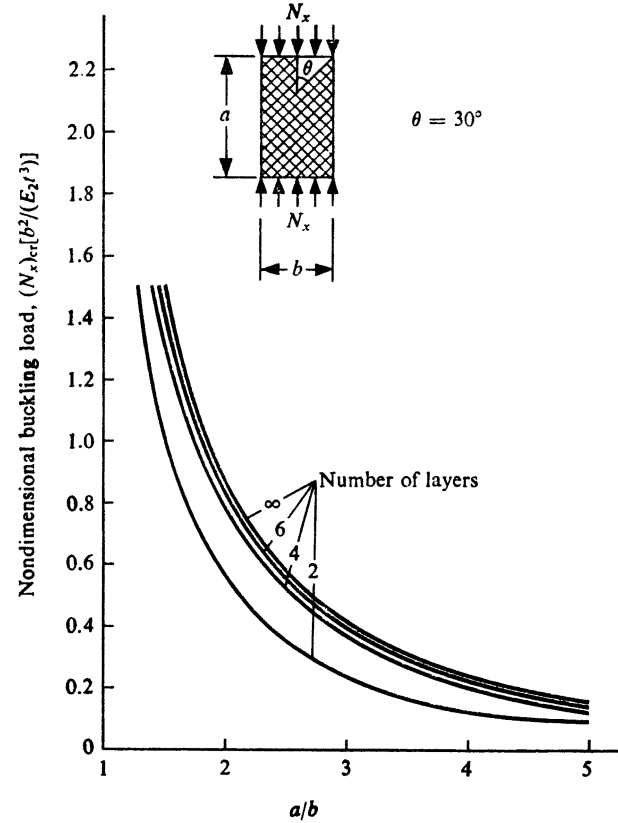


Fig. 6.9 Variation of Buckling Load with Aspect Ratio.

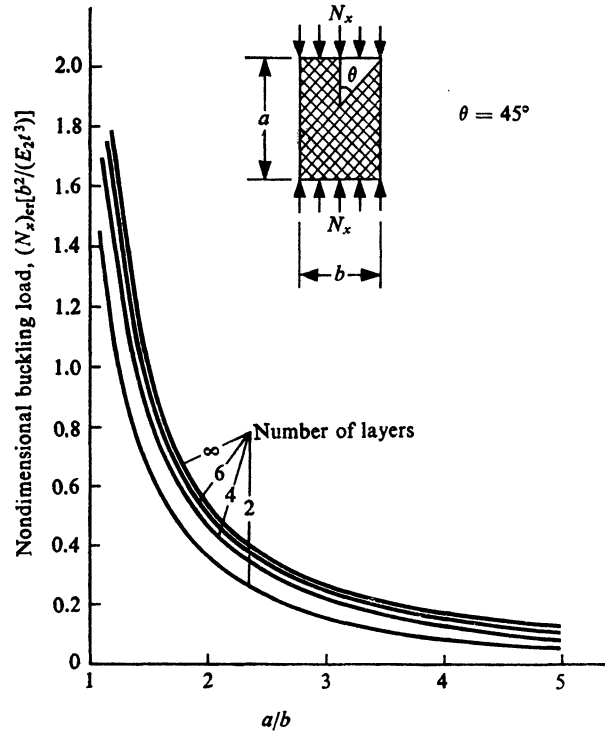


Fig. 6.10 Variation of Buckling Load with Aspect Ratio.

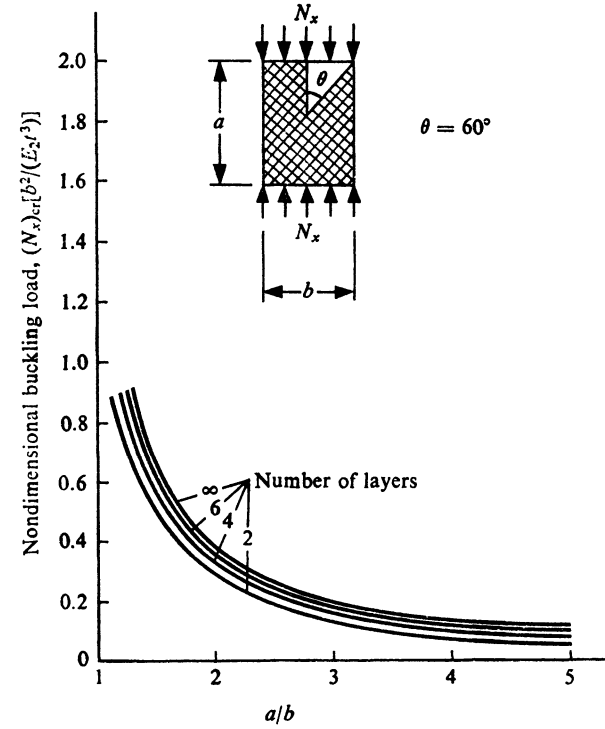


Fig. 6.11 Variation of Buckling Load with Aspect Ratio.

EXAMPLE 6.4 (Buckling of laminated composite plate with loading edges simply-supported and other two edges clamped) Consider, once again, the laminated rectangular plate in Fig. 6.5, with clamped boundary conditions along the edges $y = \pm b/2$. The governing differential equation is (6.62). The boundary conditions along $\xi = 0$, i are given by (6.63a), (6.63b), (6.63c), and (6.63d). Further, the boundary conditions along the clamped edges are

$$u^0 = B_{16}(2A_{66} - A_{12})\frac{\partial^5 \phi}{\partial \xi^4 \partial \eta} - B_{26}(2A_{66} + 3A_{12} - 3A_{11})p^2\frac{\partial^5 \phi}{\partial \xi^2 \partial \eta^3} + B_{26}A_{22}p^2\frac{\partial^5 \phi}{\partial \eta^5} = 0, \quad (6.69a)$$

$$v^0 = B_{16}A_{11}\frac{\partial^5 \phi}{\partial \xi^5} - \{B_{16}(2A_{66} + 3A_{12}) - 3A_{11}B_{26}\}p^2\frac{\partial^5 \phi}{\partial \xi^3 \partial \eta^2} + B_{26}(2A_{66} - A_{12})p^4\frac{\partial^5 \phi}{\partial \xi \partial \eta^4} = 0, \quad (6.69b)$$

$$w = A_{16}A_{66}\frac{\partial^4 \phi}{\partial \eta^4} + A_{66}A_{22}p^4\frac{\partial^4 \phi}{\partial \eta^4} + (A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2)p^2\frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} = 0, \quad (6.69c)$$

$$\frac{\partial w}{\partial \eta} = A_{16}A_{66}\frac{\partial^5 \phi}{\partial \xi^4 \partial \eta} + A_{16}A_{22}p^4\frac{\partial^5 \phi}{\partial \eta^5} + (A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2)p^2\frac{\partial^5 \phi}{\partial \xi^2 \partial \eta^3} = 0. \quad (6.69d)$$

Following the procedure described in Example 6.3, we get four simultaneous homogeneous algebraic equations from which the critical load N_x can be obtained.

We now discuss the results of the numerical computations we carried out for different values of the aspect ratio of angle-ply laminates of three different composite materials with varying number of laminae. The materials and their properties are as follows:

- (i) Boron fibre reinforced plastic (BFRP) with $E_1/E_2 = 10.0$, $G_{12}/E_2 = 0.33$, $\nu_{12} = 0.30$.
- (ii) Carbon fibre reinforced plastic (CFRP) with $E_1/E_2 = 6.33$, $G_{12}/E_2 = 0.667$, $\nu_{12} = 0.20$.
- (iii) Graphite fibre reinforced plastic (GFRP) with $E_1/E_2 = 40.0$, $G_{12}/E_2 = 0.50$, $\nu_{12} = 0.25$.

Figures 6.12, 6.13, and 6.14 show the variation of buckling load as a function of θ for an antisymmetric angle-ply square plate for the materials (i), (ii), and (iii), respectively. As can be observed, the buckling load, for each material, reaches a maximum in the region $\theta = 45^\circ$ approximately. Further, the buckling mode along the X -direction has two half sine waves beyond $\theta > 40^\circ$ unlike in Example 6.3 where the mode has one half sine wave for all values of θ . The rise in the buckling load in the region $0^\circ \leq \theta \leq 45^\circ$ is due to the increasing effect of the clamped edges as θ goes up. As a result, the effective cross stiffness increases. Also, at 0° , the buckling load is of the same order of magnitude as in Example 6.3. For an infinite number of layers, the result is the same as that for an orthotropic plate where all the coupling terms are zero.

Table 6.4 gives the percentage reduction in the buckling load between four- and two-layered, and between six- and four-layered configurations of a square antisymmetric angle-ply CFRP composite plate as a function of θ .

The variation of the buckling load as a function of the thickness ratio t_1/t_2 for a four-layered CFRP square plate with three different fibre orientations, namely, 30° , 45° , and 60° , is shown in Fig. 6.15. At each orientation, the buckling load reaches a maximum at

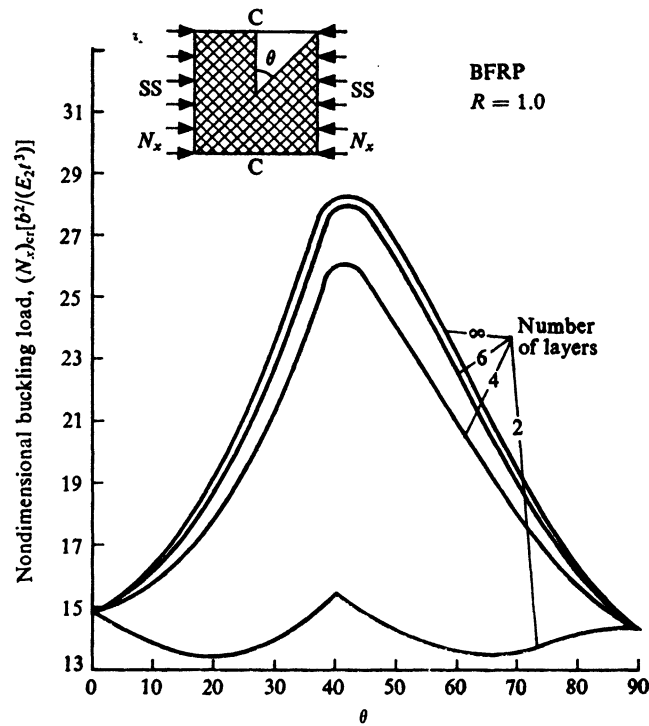


Fig. 6.12 Variation of Buckling Load with Fibre Orientation.

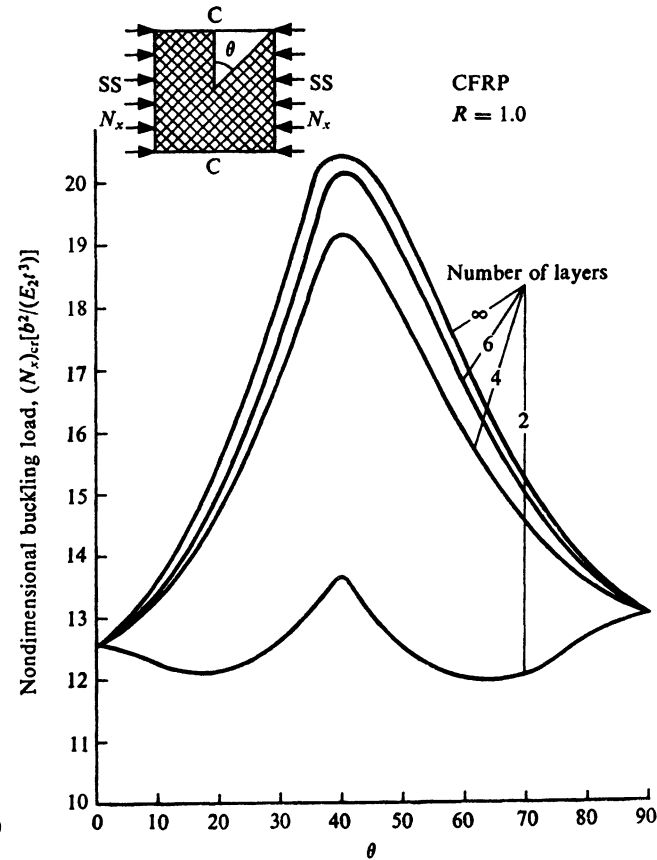


Fig. 6.13 Variation of Buckling Load with Fibre Orientation.

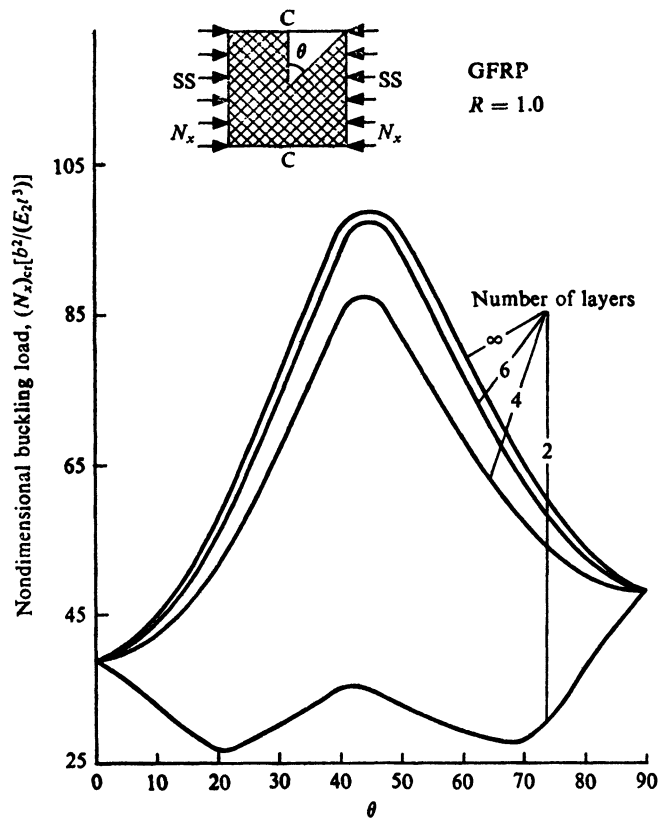


Fig. 6.14 Variation of Buckling Load with Fibre Orientation.

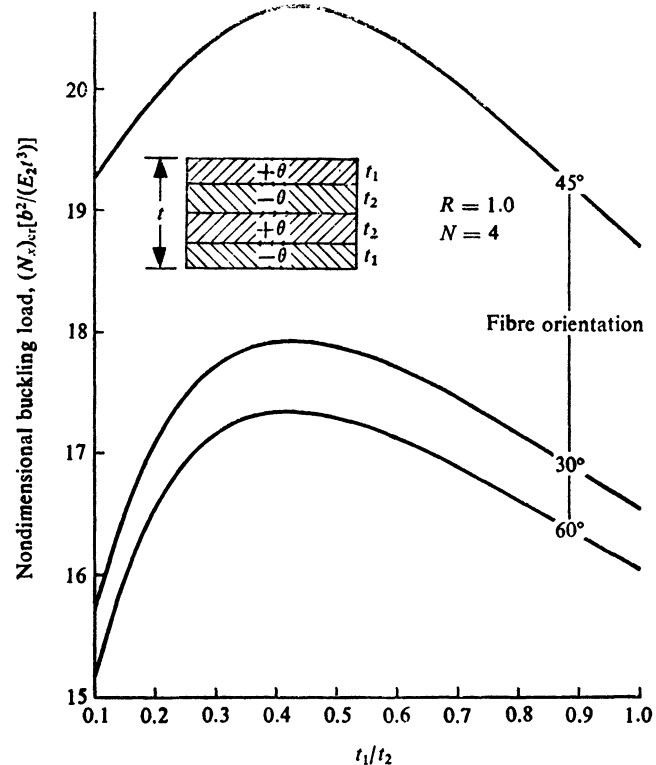


Fig. 6.15 Variation of Buckling Load with Thickness Ratio.

Table 6.4 Percentage Reduction in Buckling Load As A Consequence of Coupling for CFRP Square Antisymmetric Angle-Ply Plate

Fibre orientation (θ)	Percentage reduction between $N = 4$ and $N = 2$	Percentage reduction between $N = 6$ and $N = 4$
15	9.82	1.50
20	15.73	2.20
25	18.94	4.04
30	23.30	3.74
35	27.03	5.13
40	28.57	4.80
45	30.05	5.27
50	29.31	5.23
55	28.23	4.98
60	24.63	4.42
65	20.78	3.77
70	16.34	2.60
75	10.33	2.05

$t_1/t_2 = 0.4142$. The percentage reduction in the buckling load between an equal thickness ratio laminate ($t_1/t_2 = 1.0$) and an optimum thickness ratio laminate ($t_1/t_2 = 0.4142$) is of the order of 8 for 30° and 60° fibre orientations and 10 for 45° fibre orientation.

Figures 6.16 and 6.17 show the variation of buckling load with the aspect ratio p for GFRP and CFRP plates of varying layers with 30° fibre orientation. The cusps in these figures represent the changes in the mode shape along the X -direction as the aspect ratio increases. It is seen that the location of cusps depends on the number of layers in, and θ and degree of anisotropy of, the material.

EXAMPLE 6.5 (Buckling of composite cross-ply plate with loading edges simply-supported and other two edges free) Consider, yet again, the plate in Fig. 6.5, with lamina orientation alternating as 0° and 90°. The equation governing the plate here is similar to Eq. (6.62). However, the coefficients of A_1 , A_3 , A_5 , A_7 , A_9 are different. The governing equation is

$$\begin{aligned}
 & A_1 \frac{\partial^8 \phi}{\partial \xi^8} + A_3 p^2 \frac{\partial^8 \phi}{\partial \xi^6 \partial \eta^2} + A_5 p^4 \frac{\partial^8 \phi}{\partial \xi^4 \partial \eta^4} + A_7 p^6 \frac{\partial^8 \phi}{\partial \xi^2 \partial \eta^6} + A_9 p^8 \frac{\partial^8 \phi}{\partial \eta^8} + N_x a^2 \{ A_{11} A_{66} \frac{\partial^6 \phi}{\partial \xi^6} \\
 & + p^2 (A_{11} A_{22} - A_{12}^2 + 2 A_{12} A_{66}) \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + A_{22} A_{66} p^4 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} \} = 0, \quad (6.70)
 \end{aligned}$$

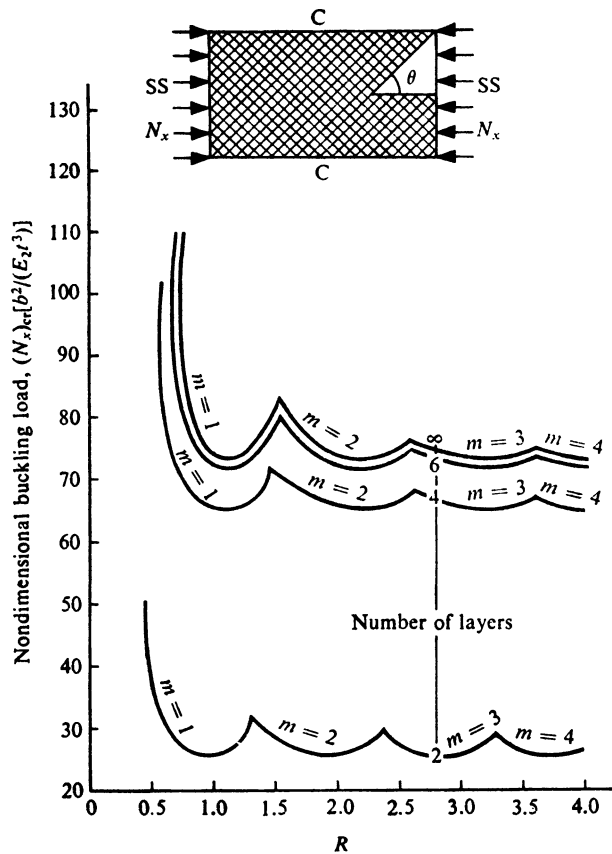


Fig. 6.16 Variation of Buckling Load with Aspect Ratio.

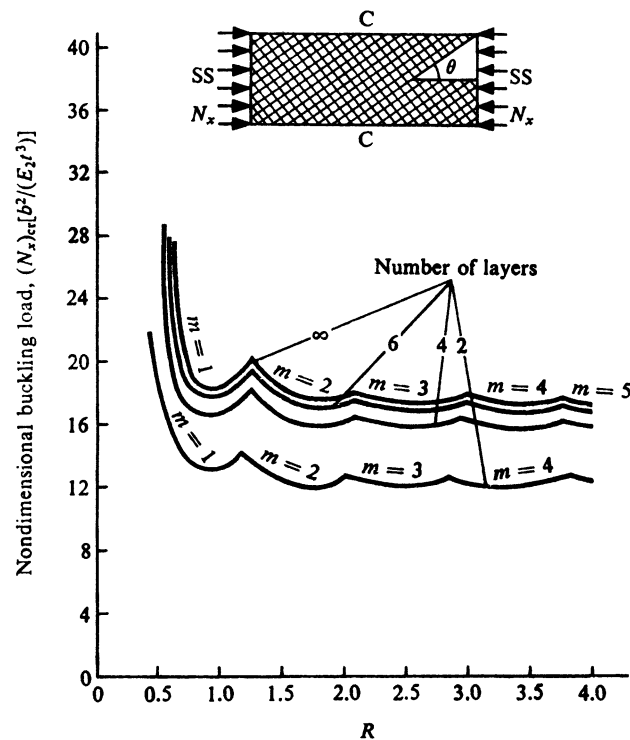


Fig. 6.17 Variation of Buckling Load with Aspect Ratio.

where

$$A_1 = A_{66}(A_{11}D_{11} - B_{11}^2),$$

$$A_3 = 2A_{11}A_{66}(D_{12} + 2D_{66}) - A_{22}B_{11}^2 + D_{11}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}),$$

$$A_5 = 2(D_{12} + 2D_{66})(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - 2B_{11}^2(A_{12} + A_{66}) \\ + A_{66}(A_{11}D_{22} + A_{22}D_{11}),$$

$$A_7 = 2A_{22}A_{66}(D_{12} + 2D_{66}) - A_{11}B_{11}^2 + D_{22}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}),$$

$$A_9 = A_{66}(A_{22}D_{22} - B_{11}^2).$$

The boundary conditions along the edges $\xi = 0, 1$ are

$$u^0 = B_{11}A_{16}\frac{\partial^5\phi}{\partial\xi^5} + B_{11}A_{22}p^2\frac{\partial^5\phi}{\partial\xi^3\partial\eta^2} + B_{11}(A_{12} + A_{66})p^4\frac{\partial^5\phi}{\partial\xi\partial\eta^4} = 0, \quad (6.71a)$$

$$w = A_{11}A_{66}\frac{\partial^4\phi}{\partial\xi^4} + A_{66}A_{22}p^4\frac{\partial^4\phi}{\partial\eta^4} + (A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2)p^2\frac{\partial^4\phi}{\partial\xi^2\partial\eta^2} = 0, \quad (6.71b)$$

$$M_\xi = b_{11}\frac{\partial^6\phi}{\partial\xi^6} + b_{12}p^2\frac{\partial^6\phi}{\partial\xi^4\partial\eta^2} + b_{13}p^4\frac{\partial^6\phi}{\partial\xi^2\partial\eta^4} + b_{14}p^6\frac{\partial^6\phi}{\partial\eta^6} = 0, \quad (6.71c)$$

$$N_{\xi\eta} = b_{15}\frac{\partial^6\phi}{\partial\xi^5\partial\eta} + b_{16}p^2\frac{\partial^6\phi}{\partial\xi^3\partial\eta^3} + b_{17}p^4\frac{\partial^6\phi}{\partial\xi\partial\eta^5}, \quad (6.71d)$$

and those along the edges $\eta = \pm 1/2$ are

$$N_\eta = b_{18}\frac{\partial^6\phi}{\partial\xi^6} + b_{19}p^2\frac{\partial^6\phi}{\partial\xi^4\partial\eta^2} + b_{20}p^4\frac{\partial^6\phi}{\partial\xi^2\partial\eta^4} = 0, \quad (6.71e)$$

$$N_{\xi\eta} = b_{15}\frac{\partial^6\phi}{\partial\xi^5\partial\eta} + b_{16}p^2\frac{\partial^6\phi}{\partial\xi^3\partial\eta^3} + b_{17}p^4\frac{\partial^6\phi}{\partial\xi\partial\eta^5} = 0, \quad (6.71f)$$

$$M_\eta = b_{21}\frac{\partial^6\phi}{\partial\xi^6} + b_{22}p^2\frac{\partial^6\phi}{\partial\xi^4\partial\eta^2} + b_{23}p^4\frac{\partial^6\phi}{\partial\xi^2\partial\eta^4} + b_{24}p^6\frac{\partial^6\phi}{\partial\eta^6} = 0, \quad (6.71g)$$

$$2\frac{\partial M_{\xi\eta}}{\partial\xi} + p\frac{\partial M_\eta}{\partial\eta} = b_{25}\frac{\partial^7\phi}{\partial\xi^6\partial\eta} + b_{26}p^2\frac{\partial^7\phi}{\partial\xi^4\partial\eta^3} + b_{27}p^4\frac{\partial^7\phi}{\partial\xi^2\partial\eta^5} + b_{24}p^6\frac{\partial^7\phi}{\partial\eta^7} = 0, \quad (6.71h)$$

where

$$b_{11} = A_{66}(B_{11}^2 - A_{11}D_{11}),$$

$$b_{12} = B_{11}^2A_{22} - D_{11}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - A_{11}A_{66}D_{12},$$

$$b_{13} = B_{11}^2(A_{12} + A_{66}) - A_{22}A_{66}D_{11} - D_{12}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}),$$

$$b_{14} = -A_{22}A_{66}D_{12},$$

$$b_{15} = -A_{12}A_{66}B_{11},$$

$$b_{16} = B_{11}A_{66}(A_{22} - A_{11}),$$

$$b_{17} = A_{12}A_{66}B_{11},$$

$$b_{18} = A_{12}A_{16}B_{11},$$

$$b_{19} = b_{16},$$

$$b_{20} = -B_{11}A_{12}A_{66},$$

$$b_{21} = -A_{11}A_{66}D_{12},$$

$$b_{22} = B_{11}^2(A_{12} + A_{66}) - D_{12}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}),$$

$$b_{23} = B_{11}^2A_{11} - D_{12}A_{22}A_{66} - D_{22}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}),$$

$$b_{24} = A_{66}(B_{11}^2 - A_{22}D_{22}),$$

$$b_{25} = -A_{11}A_{66}(4D_{66} + D_{12}),$$

$$b_{26} = B_{11}^2(A_{12} + A_{66}) - D_{12}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - D_{22}A_{11}A_{66},$$

$$b_{27} = B_{11}^2A_{11} - 4D_{66}A_{22}A_{66} - D_{12}A_{22}A_{66} - D_{22}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}).$$

We now discuss the results of the numerical computations we carried out for different values of the aspect ratio of a CFRP antisymmetric cross-ply laminate with varying number of laminae.

Figure 6.18 shows the variation of nondimensional buckling load as a function of the aspect ratio for a regular ($M = 1$) antisymmetric cross-ply laminate. The dependence of the buckling load progressively decreases with increase in the number of layers. The variation of buckling load for a square antisymmetric cross-ply plate with cross-ply ratio M is shown in Fig. 6.19. It is seen that the buckling load increases with M , the increase being maximum between cross-ply ratios of 1 and 3, whereafter the buckling load becomes asymptotic with cross-ply ratio. For a designer, this may indicate the optimal cross-ply ratio beyond which an increase in this ratio becomes unnecessary to achieve a given buckling load.

6.4 APPROXIMATE TECHNIQUES

The closed-form solution of the differential equations derived in Sections 6.2 and 6.3 is possible only when the boundary conditions are simple and for uniform inplane loadings. What is more, even for the simple case of a plate with all the four edges clamped, no closed-form solution is possible. In such situations, an attempt can be made to apply one of the approximate techniques which have the inherent ability to converge to the exact solution.

6.4.1 Galerkin's Technique

As we have already noted, the Galerkin technique yields an approximate buckling load. Further, in a situation where it is not possible to obtain a closed-form solution through the governing differential equation, we can obtain an approximate solution by using the Galerkin

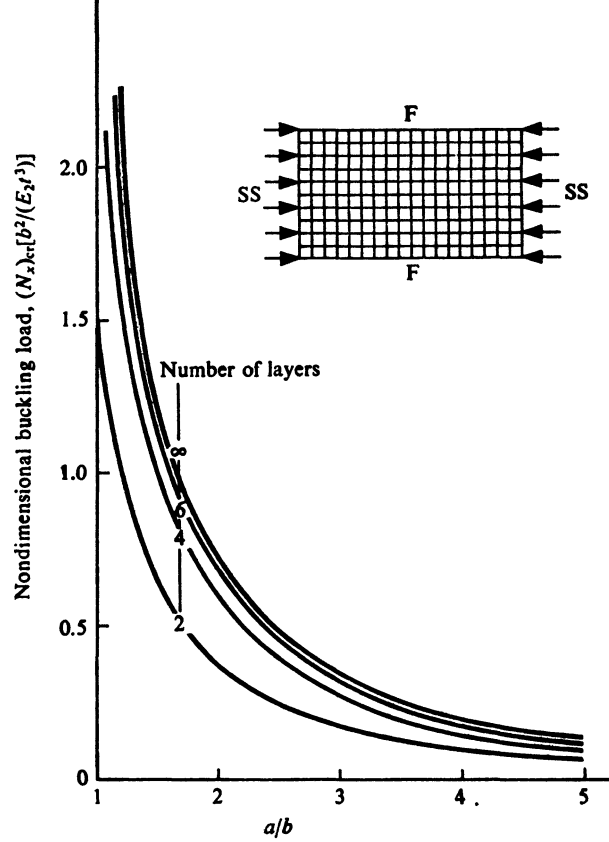


Fig. 6.18 Variation of Buckling Load with Aspect Ratio.

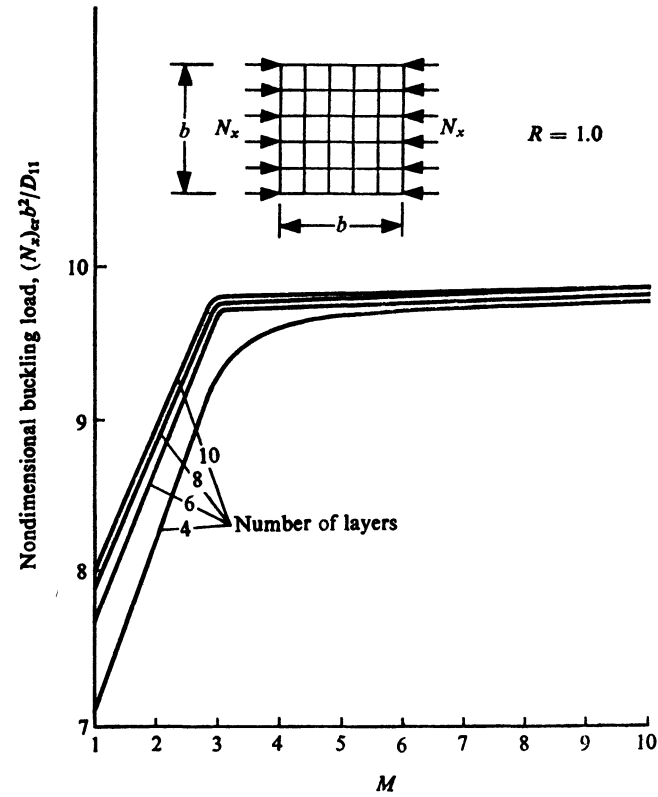


Fig. 6.19 Variation of Buckling Load with Cross-Ply Ratio.

integral (1.195) which is true for a one-dimensional structure. In Section 5.4.2, we extended this method to two-dimensional structures, i.e., plates, and obtained the Galerkin equation, i.e., (5.106). This equation is valid only for isotropic plates. In this section, we shall demonstrate the application of the Galerkin technique to composite plates.

EXAMPLE 6.6 (Buckling of simply-supported angle-ply composite plate subjected to varying uniaxial load) In Examples 6.4 and 6.5, we assumed that the inplane end load is uniform. Such a situation is rather rare in practice. Here, we shall assume that the end load varies as

$$N_x = N_0(1 - \alpha\eta), \quad (6.72)$$

where N_0 is the intensity of compressive load at the edges $\eta = 0, 1$ and α is a numerical factor. $\alpha = 0$ corresponds to uniform loading and $\alpha = 2$ to pure bending; for any other value of α , the loading corresponds to bending and compression.

Substituting the expression for N_x in Eq. (6.62), we get the governing equation as

$$\begin{aligned} & A_1 \frac{\partial^8 \phi}{\partial \xi^8} + A_3 p^2 \frac{\partial^8 \phi}{\partial \xi^6 \partial \eta^2} + A_5 p^4 \frac{\partial^8 \phi}{\partial \xi^4 \partial \eta^4} + A_7 p^6 \frac{\partial^8 \phi}{\partial \xi^2 \partial \eta^6} + A_9 p^8 \frac{\partial^8 \phi}{\partial \eta^8} \\ & + N_0(1 - \alpha\eta) a^2 \{ A_{16} A_{66} \frac{\partial^6 \phi}{\partial \xi^6} + p^2 (A_{11} A_{22} - 2A_{12} A_{66} - A_{12}^2) \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} \\ & + A_{22} A_{66} p^4 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} \} = 0. \end{aligned} \quad (6.73)$$

The boundary conditions along the edges $\xi = 0, 1$ are given by (6.63a), (6.63b), (6.63c), and (6.63d). Further, along the edges $\eta = 0, 1$, the boundary conditions are

$$\begin{aligned} v^0 &= B_{16} A_{11} \frac{\partial^3 \phi}{\partial \xi^3} - \{ B_{16} (2A_{66} + 3A_{12}) - 3A_{11} B_{26} \} p^2 \frac{\partial^3 \phi}{\partial \xi^3 \partial \eta^2} \\ &+ B_{26} (2A_{66} - A_{12}) p^4 \frac{\partial^3 \phi}{\partial \xi \partial \eta^4} = 0, \end{aligned} \quad (6.74a)$$

$$w = A_{16} A_{66} \frac{\partial^4 \phi}{\partial \xi^4} + A_{66} A_{22} p^4 \frac{\partial^4 \phi}{\partial \eta^4} + (A_{11} A_{22} - 2A_{12} A_{66} - A_{12}^2) p^2 \frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} = 0, \quad (6.74b)$$

$$M_\eta = a_{11} \frac{\partial^6 \phi}{\partial \xi^6} + a_{12} p^2 \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{13} p^4 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} + a_{14} p^6 \frac{\partial^6 \phi}{\partial \eta^6} = 0, \quad (6.74c)$$

$$M_{\xi\eta} = a_{15} \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{16} p^2 \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} = 0, \quad (6.74d)$$

where

$$a_{11} = A_{11} (B_{16} B_{26} - A_{66} D_{12}),$$

$$a_{12} = A_{11} (3B_{26}^2 - A_{66} D_{22}) - 4B_{16} B_{26} A_{12} - D_{12} (A_{11} A_{22} - A_{12}^2 - 2A_{12} A_{66}),$$

$$a_{13} = A_{22} (3B_{16} B_{26} - D_{12} A_{66}) - 4B_{26}^2 A_{12} - D_{12} (A_{11} A_{22} - A_{12}^2 - 2A_{12} A_{66}),$$

$$a_{14} = A_{22}(B_{26}^2 - D_{22}A_{66}),$$

$$a_{15} = 2B_{26}A_{11}A_{66} + B_{16}(A_{12}^2 - 2A_{12}A_{66} - A_{11}A_{22}),$$

$$a_{16} = 2B_{16}A_{22}A_{66} + B_{26}(A_{12}^2 - 2A_{12}A_{66} - A_{11}A_{22}).$$

Equation (6.73) is a variable coefficient differential equation, and hence it is difficult to obtain its closed-form solution. We shall therefore analyze the plate we are considering[†] by applying the Galerkin technique (see Section 5.4.2).

A function of the type

$$\phi(\xi, \eta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi\xi \sin n\pi\eta \quad (6.75)$$

satisfies the boundary conditions. Now, applying the Galerkin technique, the resulting integral we get is

$$\begin{aligned} & \int_0^1 \int_0^1 \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} A_{mn} \sin m\pi\xi \sin n\pi\eta \right) \sin p\pi\xi \sin q\pi\eta \, d\xi \, d\eta \\ & - N_0 \int_0^1 \int_0^1 \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} A_{mn} \sin m\pi\xi \sin n\pi\eta \right) \sin p\pi\xi \sin q\pi\eta \, d\xi \, d\eta \\ & + N_0 \alpha \int_0^1 \int_0^1 \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} A_{mn} \eta \sin m\pi\xi \sin n\pi\eta \right) \sin p\pi\xi \sin q\pi\eta \, d\xi \, d\eta \\ & = 0 \quad (p, q = 1, 2, \dots), \end{aligned} \quad (6.76)$$

where

$$C_{mn} = \pi^8 (A_1 m^8 + A_3 m^6 n^2 + A_5 m^4 n^4 + A_7 m^2 n^6 + A_9 n^8),$$

$$D_{mn} = \pi^6 [A_{16} A_{66} m^6 + p^2 (A_{11} A_{22} - 2A_{12} A_{66} - A_{12}^2) m^4 n^2 + A_{22} A_{66} p^4 n^6].$$

Making use of the identity

$$\begin{aligned} \int_0^1 \sin i\pi\xi \sin j\pi\xi \, d\xi &= 0 \quad \text{for } i \neq j \\ &= 1/2 \quad \text{for } i = j, \end{aligned}$$

we find Eq. (6.75) reduces to

$$\begin{aligned} \frac{1}{4} C_{pq} A_{pq} - \frac{N_0}{4} D_{pq} A_{pq} + N_0 \frac{\alpha}{2} \left(\sum_{i=1}^{\infty} D_{pi} A_{pi} I_{qi} \right) &= 0 \quad (p = 1, 2, \dots, m, \\ & q = 1, 2, \dots, n), \end{aligned} \quad (6.77)$$

where

$$I_{qn} = \int_0^1 \eta \sin n\pi\eta \sin q\pi\eta \, d\eta$$

$$= \frac{1}{2}$$
$$= 0$$
$$= -\frac{4}{\pi^2} \frac{nq}{(n^2 - q^2)^2}$$

$$(q = n)$$
$$(q \neq n, q \pm n \text{ is an even number})$$
$$(q \neq n, q \pm n \text{ is an odd number}).$$

This leads to an infinite set of homogeneous equations for A_{pq} . In such a situation, N_0 is determined by the process of iteration for a given value of α .

We now discuss the results of the numerical computations we carried out for a square angle-ply laminate of two different materials with varying number of laminae.

Figure 6.20 shows the variation of buckling load with fibre orientation for a square GFRP laminate with $\alpha = 2$. As can be observed, the results here are similar to those in Examples 6.3 and 6.4.

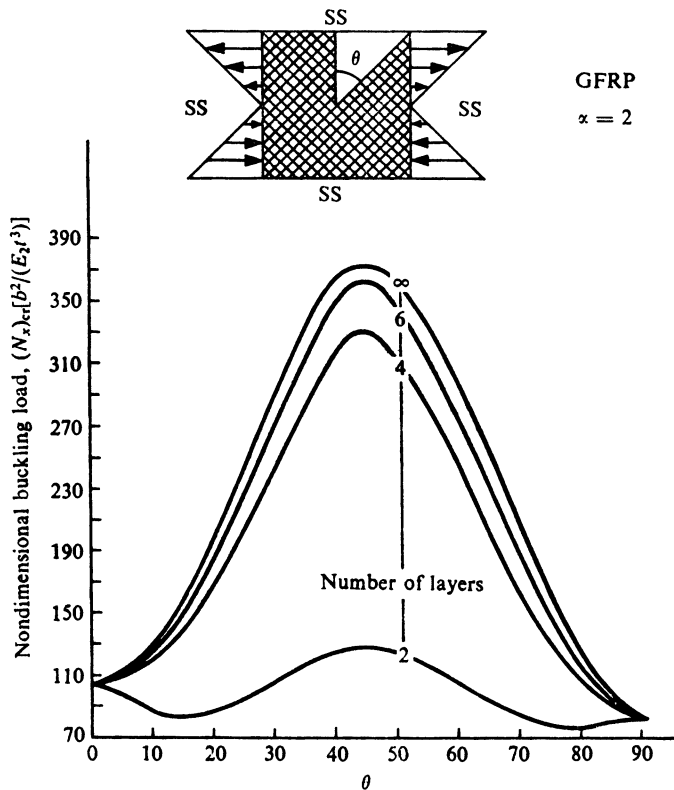


Fig. 6.20 Variation of Buckling Load with Fibre Orientation.

Figure 6.21 shows the variation of buckling load with the thickness ratio t_1/t_2 for a four-ply laminate for two different loading conditions (i.e., $\alpha = 0, 2$). A comparison for the two

loading conditions shows that, for each load, the buckling load is maximum when the fibre orientation is 45° . However, the percentage increase is slightly more when $\alpha = 0$ than when $\alpha = 2$.

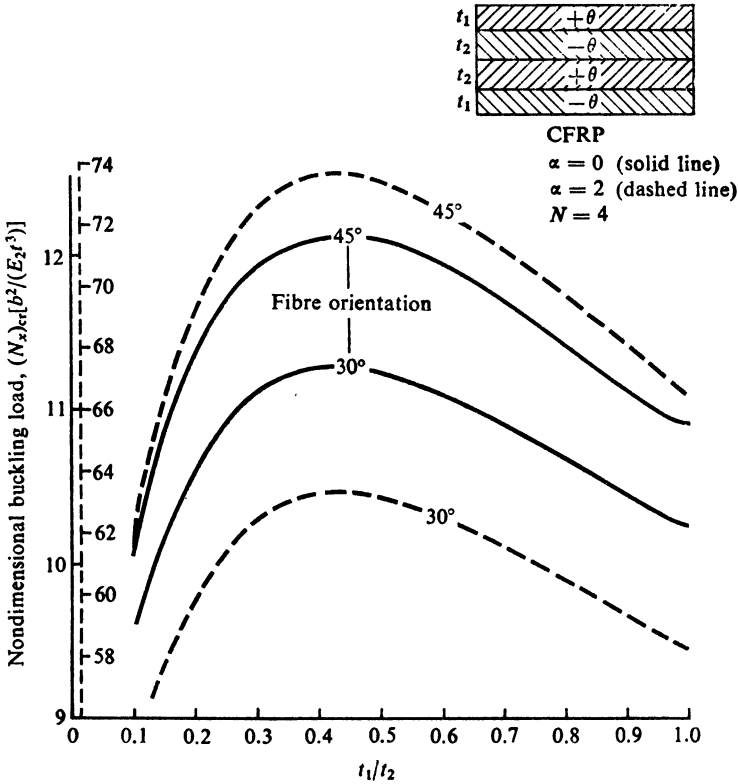


Fig. 6.21 Variation of Buckling Load with Thickness Ratio.

PROBLEMS

6.1 Obtain the buckling load of an antisymmetric cross-ply laminate, simply-supported on all sides and subjected to an axial compressive loading. Plot the variation of the buckling load with the aspect ratio of the plate, taking the number of layers as the parameter, for the following composite materials:

- (i) Graphite-epoxy ($E_1/E_2 = 40$, $G_{12}/E_2 = 0.5$, $\nu_{12} = 0.25$).
- (ii) Carbon-epoxy ($E_1/E_2 = 6.33$, $G_{12}/E_2 = 0.667$, $\nu_{12} = 0.20$).
- (iii) Boron-epoxy ($E_1/E_2 = 10$, $G_{12}/E_2 = 0.33$, $\nu_{12} = 0.30$).

6.2 Plot the buckling load versus the aspect ratio for a specially orthotropic rectangular

plate simply-supported along the loaded edges and clamped along the other two edges. The plate is subjected to a uniaxial compressive loading.

6.3 How would the buckling load of the plate in Problem 6.2 be affected if the plate is subjected to a biaxial loading?

6.4 Derive the equilibrium equation for a skew, rectangular, laminated plate subjected to a uniaxial loading.

6.5 An angle-ply laminated clamped rectangular plate of graphite-epoxy is subjected to a uniaxial compressive loading. Obtain the variation of buckling load with aspect ratio.

6.6 A symmetric laminated plate is made up of four laminae. The top and the bottom layers are of carbon-epoxy and the central layers are of glass-epoxy. What will be the buckling load of such a hybrid laminate if the laminate is subjected to a uniform, uniaxial compressive load? Assume the plate is simply-supported on all sides.

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Inelastic Buckling of Plates

7.1 INTRODUCTION

In Chapters 5 and 6, we studied the buckling of plates, assuming Hooke's law. Further, in Chapter 2, we showed that in a column the stresses may, depending on the size and shape of the column, exceed the elastic limit before the buckling stress is reached. In a plate too the stresses may go beyond the elastic limit before buckling occurs, but here this depends on the plate thickness and the load applied. Unlike in a column, there exist two loading axes in a plate. If the load is applied only along the longitudinal direction, then the stress in this direction is likely to exceed the elastic limit and that in the transverse direction may remain within the elastic limit. Here, the plate response is governed by two moduli. If, however, the load is acting on both the axes, the entire plate will exhibit inelastic behaviour. In either situation, the elastic theory (see Chapter 5) has to be modified before it is applied. This can be done by so correcting the constitutive equation that it takes into account the inelastic effect. The theory Stowell [1] developed for inelastic buckling is based on the plasticity hypothesis and uses incremental constitutive equations. Since this theory is quite involved and requires the knowledge of tangent and secant modulus of the plate material, Bleich [2] proposed a simplified version of the theory. According to him, when the stress σ_x along the X -axis exceeds the elastic limit, the tangent modulus E_t governs the deformation along this axis, and the elastic modulus E governs the deformation along the Y -direction. Further, when the stress exceeds the elastic limit in both the directions, then E_t governs the deformation in both the X - and Y -direction.

In what follows, we shall discuss the two approaches in detail.

7.2 STOWELL'S THEORY

The Stowell theory is based on Shanley's approach for inelastic columns, where it is assumed that there is no strain reversal.

In our description here, we shall take as valid (i) the two assumptions we made in Section 5.2 on the behaviour of a thin plate, (ii) the basic force and moment equilibrium equations, namely, (5.5), (5.6), (5.17), and (5.18), and (iii) the strain-displacement relations (5.22). However, the change occurs only in Eqs. (5.20).

7.2.1 Inelastic Constitutive Equations

The derivation we shall present here closely follows that given in Gerard [3] and Rivello [4].

Since we have assumed that the plate is thin, only the inplane stresses σ_x , σ_y , and τ_{xy} exist and all the other stresses are zero. To simplify the analysis, it is advisable to represent the complex state of stress or strain by means of the invariant functions of stress or strain. Examples of such invariants are octahedral shear stress or shear strain, and *effective* or *significant* stress or strain. For a two-dimensional state of stress, the effective stress and strain are defined as

$$\begin{aligned}\bar{\sigma} &= (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2)^{1/2}, \\ \bar{\epsilon} &= \frac{2}{\sqrt{3}}(\epsilon_x^2 + \epsilon_y^2 + \epsilon_x \epsilon_y - \frac{\gamma_{xy}^2}{4})^{1/2}.\end{aligned}\quad (7.1)$$

Here, $\bar{\sigma}$ may be treated as an equivalent tensile stress, producing the same strain $\bar{\epsilon}$ as the combined stresses σ_x , σ_y , and τ_{xy} . $\bar{\sigma}$ is uniquely defined by a function of $\bar{\epsilon}$ as

$$\bar{\sigma} = \phi(\bar{\epsilon}), \quad (7.2)$$

where ϕ is called the plasticity function.

We shall make the following assumptions for deriving the constitutive equations:

- (i) The plate material is continuous and isotropic.
- (ii) The principal axes of plastic stress and strain coincide at all times.
- (iii) The volume of the material remains constant. This assumption requires that Poisson's ratio must increase from its elastic value (≈ 0.3) to the value 0.5 for the plastic condition.

In view of our assumptions, the following relations between stress and strain hold:

$$\begin{aligned}\frac{\sigma_x - \nu \sigma_y}{\epsilon_x} &= \frac{\bar{\sigma}}{\bar{\epsilon}} = \frac{\phi(\bar{\epsilon})}{\bar{\epsilon}}, \\ \frac{\sigma_y - \nu \sigma_x}{\epsilon_y} &= \frac{\sigma}{\bar{\epsilon}} = \frac{\phi(\bar{\epsilon})}{\bar{\epsilon}}, \\ \frac{\tau_{xy}}{2(1 + \nu)\gamma_{xy}} &= \frac{\bar{\sigma}}{\bar{\epsilon}} = \frac{\phi(\bar{\epsilon})}{\bar{\epsilon}}.\end{aligned}\quad (7.3)$$

The plasticity function is obtained from the stress-strain curve of the material. Since $\bar{\sigma}/\bar{\epsilon} = E_s$ (secant modulus), $\phi(\bar{\epsilon})/\bar{\epsilon}$ is also equal to E_s . Hence, the constitutive equations (7.3) become

$$\epsilon_x = \frac{1}{E_s}(\sigma_x - \frac{1}{2}\sigma_y), \quad (7.4a)$$

$$\epsilon_y = \frac{1}{E_s}(\sigma_y - \frac{1}{2}\sigma_x), \quad (7.4b)$$

$$\gamma_{xy} = \frac{3}{E_s}\tau_{xy}. \quad (7.4c)$$

Relations (7.4) can also be written in terms of the stresses as

$$\sigma_x = \frac{4}{3}E_s(\epsilon_x + \frac{1}{2}\epsilon_y), \quad (7.5a)$$

$$\sigma_y = \frac{4}{3}E_s(\epsilon_y + \frac{1}{2}\epsilon_x), \quad (7.5b)$$

$$\tau_{xy} = \frac{E_s}{3}\gamma_{xy}. \quad (7.5c)$$

7.2.2 Governing Equation

As a result of the change in the constitutive equations in the plastic region, the moment-displacement relations also change. Thus, up to the point of buckling, the only stress acting in the plate is σ_x . Hence, from Eq. (7.5b),

$$\epsilon_y = -\frac{1}{2}\epsilon_x. \quad (7.6)$$

Substituting relation (7.6) in Eq. (7.5a), we get

$$\sigma_x = E_s\epsilon_x. \quad (7.7)$$

After buckling, the distortion of the plate gives rise to a variation in the strains ϵ_x , ϵ_y , and γ_{xy} . This distortion is mainly due to the bending of the plate. Hence, from Eq. (7.5a), we have

$$\delta\sigma_x = \frac{4}{3}E_s(\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y) + \frac{4}{3}(\epsilon_x + \frac{1}{2}\epsilon_y)\delta E_s. \quad (7.8)$$

Further, from Eq. (7.7), we obtain

$$\begin{aligned} \delta E_s &= \delta\left(\frac{\sigma_x}{\epsilon_x}\right) = \frac{1}{\epsilon_x}\delta\sigma_x - \frac{\sigma_x}{\epsilon_x^2}\delta\epsilon_x \\ &= \frac{\delta\epsilon_x}{\epsilon_x}\left(\frac{\delta\sigma_x}{\delta\epsilon_x} - \frac{\sigma_x}{\epsilon_x}\right). \end{aligned}$$

Identifying $\delta\sigma_x/(\delta\epsilon_x) = E_t$ (tangent modulus) and $\sigma_x/\epsilon_x = E_s$, we find Eq. (7.8) becomes

$$\delta E_s = \frac{\delta\epsilon_x}{\epsilon_x}(E_t - E_s). \quad (7.9)$$

Substituting Eq. (7.9) in Eq. (7.8) and simplifying, we get

$$\delta\sigma_x = \frac{4}{3}E_s(\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y) + (E_t - E_s)\delta\epsilon_x. \quad (7.10)$$

Equation (7.10) can be rewritten as

$$\delta\sigma_x = \frac{4}{3}E_s\left[\left(\frac{1}{2} + \frac{3}{4}\frac{E_t}{E_s}\right)\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y\right]. \quad (7.11)$$

A variation in the strain causes a change in the curvature. Thus,

$$\delta\epsilon_x = -z\delta\chi_1, \quad \delta\epsilon_y = -z\delta\chi_2, \quad \delta\gamma_{xy} = -2z\delta\chi_3. \quad (7.12)$$

The stresses $\delta\sigma_x$, $\delta\sigma_y$, and $\delta\tau_{xy}$ give rise to the couples δM_x , δM_y , and δM_{xy} :

$$\begin{aligned}\delta M_x &= \int_{-l/2}^{l/2} \delta\sigma_x z \, dz, \\ \delta M_y &= \int_{-l/2}^{l/2} \delta\sigma_y z \, dz, \\ \delta M_{xy} &= \int_{-l/2}^{l/2} \delta\tau_{xy} z \, dz.\end{aligned}\tag{7.13}$$

Substituting for $\delta\sigma_x$ from Eq. (7.11) and integrating, we get

$$\delta M_x = -E_s \frac{l^3}{9} \left[\left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s} \right) \delta\chi_1 + \frac{1}{2} \delta\chi_2 \right].\tag{7.14}$$

To determine the incremental moment δM_y , we proceed in a manner similar to that for δM_x . Thus, we first obtain

$$\delta\sigma_y = \frac{4}{3} E_s (\delta\epsilon_y + \frac{1}{2} \delta\epsilon_x) + \frac{4}{3} (\epsilon_y + \frac{1}{2} \epsilon_x) \delta E_s.\tag{7.15}$$

Then, substituting for δE_s from Eq. (7.9), we get

$$\delta\sigma_y = \frac{4}{3} E_s (\delta\epsilon_y + \frac{1}{2} \delta\epsilon_x) + \frac{4}{3} (\epsilon_y + \frac{1}{2} \epsilon_x) (E_t - E_s) \frac{\delta\epsilon_y}{\epsilon_x}.\tag{7.16}$$

Now making use of Eq. (7.5b), we find Eq. (7.16) gets simplified to

$$\delta\sigma_y = \frac{4}{3} E_s (\delta\epsilon_y + \frac{1}{2} \delta\epsilon_x) + \frac{\sigma_y}{E_s} (E_t - E_s) \frac{\delta\epsilon_x}{\epsilon_x}.\tag{7.17}$$

Since σ_x is the only applied stress, Eq. (7.17) reduces to

$$\delta\sigma_y = \frac{4}{3} E_s (\delta\epsilon_y + \delta\epsilon_x).\tag{7.18}$$

Hence, δM_y from Eqs. (7.13), after substituting for $\delta\sigma_y$ from Eq. (7.18), is

$$\delta M_y = -E_s \frac{l^3}{9} (\delta\chi_2 + \frac{1}{2} \delta\chi_1).\tag{7.19}$$

The increment in the twisting moment δM_{xy} is obtained, starting from Eq. (7.5c), as follows. The change in τ_{xy} is

$$\delta\tau_{xy} = \frac{E_s}{3} \delta\gamma_{xy} + \frac{\gamma_{xy}}{3} \delta E_s.\tag{7.20}$$

Since $\tau_{xy} = 0$, we get

$$\delta\tau_{xy} = \frac{E_s}{3} \gamma_{xy}.\tag{7.21}$$

Substituting $\delta\tau_{xy}$ in the last of Eqs. (7.13) and simplifying, we obtain

$$\delta M_{xy} = -\frac{E_s l^3}{18} \delta\chi_3.\tag{7.22}$$

From Eq. (5.19), the equilibrium equation for an element of a plate is

$$\frac{\partial^2}{\partial x^2}(\delta M_x) + \frac{\partial^2}{\partial y^2}(\delta M_y) - 2 \frac{\partial^2}{\partial x \partial y}(\delta M_{xy}) + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0. \quad (7.23)$$

The changes in the curvature are related to deflection by the relations

$$\delta \kappa_1 = \frac{\partial^2 w}{\partial x^2}, \quad \delta \kappa_2 = \frac{\partial^2 w}{\partial y^2}, \quad \delta \kappa_3 = \frac{\partial^2 w}{\partial x \partial y}. \quad (7.24)$$

Hence,

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(\delta M_x) &= -E_s \frac{t^3}{9} \left[\left(\frac{1}{2} + \frac{3}{4} \frac{E_t}{E_s} \right) \frac{\partial^4 w}{\partial x^4} + \frac{1}{2} \frac{\partial^4 w}{\partial x^2 \partial y^2} \right], \\ \frac{\partial^2}{\partial y^2}(\delta M_y) &= -E_s \frac{t^3}{9} \left[\frac{\partial^4 w}{\partial y^4} + \frac{1}{2} \frac{\partial^4 w}{\partial x^2 \partial y^2} \right], \\ \frac{\partial^2}{\partial x \partial y}(\delta M_{xy}) &= -E_s \frac{t^3}{18} \left[\frac{\partial^4 w}{\partial x^2 \partial y^2} \right]. \end{aligned} \quad (7.25)$$

Substituting Eqs. (7.25) in Eq. (7.23) with $N_y = N_{xy} = 0$, we get

$$\left(\frac{1}{2} + \frac{3}{4} \frac{E_t}{E_s} \right) \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2} = 0, \quad (7.26)$$

where $D = E_s t^3/9$. Equation (7.26) is the governing equation for the plastic buckling of a rectangular plate loaded on two opposite sides by only a uniformly distributed load N_x . This equation has to be suitably modified for any other loading condition. For example, if only N_y is acting, we can show, after going through a procedure similar to that used for obtaining Eq. (7.26), that the governing equation becomes

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \left(\frac{1}{2} + \frac{3}{4} \frac{E_t}{E_s} \right) \frac{\partial^4 w}{\partial y^4} + \frac{N_y}{D} \frac{\partial^2 w}{\partial y^2} = 0 \quad (7.27)$$

with

$$\begin{aligned} M_x &= -D \left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} \right), \\ M_y &= -D \left[\frac{1}{2} \frac{\partial^2 w}{\partial x^2} + \left(\frac{1}{2} + \frac{3}{4} \frac{E_t}{E_s} \right) \frac{\partial^2 w}{\partial y^2} \right], \\ M_{xy} &= -D \frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \quad (7.28)$$

We shall demonstrate the application of the foregoing equations through an example.

EXAMPLE 7.1 (Buckling load of inelastic rectangular plate simply-supported on all sides) Consider a rectangular plate (Fig. 5.6) subjected to a uniform axially compressive force N_x per unit length along the edges $x = 0, a$. The governing equation is given by (7.26). In terms of the nondimensional parameters ξ and η , defined as

$$x = a\xi, \quad y = b\eta, \quad (7.29)$$

Eq. (7.26) can be written as

$$\left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) \frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \bar{\lambda}^2 \frac{\partial^2 w}{\partial \xi^2} = 0, \quad (7.30)$$

where $p = a/b$ and $\bar{\lambda}^2 = N_x a^2 / D$. Since all the edges are simply-supported, the boundary conditions in the nondimensional form are

$$w = \left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{2} p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\xi = 0, 1), \quad (7.31a)$$

$$w = \left(p^2 \frac{\partial^2 w}{\partial \eta^2} + \frac{1}{2} \frac{\partial^2 w}{\partial \xi^2}\right) = 0 \quad (\eta = 0, 1). \quad (7.31b)$$

Since $w = 0$ along $\xi = 0, 1$ and $\eta = 0, 1$,

$$\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\xi = 0, 1),$$

$$\frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\eta = 0, 1).$$

Making use of these relations, we find the boundary conditions (7.31a) and (7.31b) reduce to

$$w = \frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\xi = 0, 1), \quad (7.32a)$$

$$w = \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0, 1). \quad (7.32b)$$

These are identical to those we obtained in Example 5.1 [see Eqs. (5.45) and (5.46)]. Thus, for the plate we are considering, the boundary conditions are the same as those for elastic analysis. We see that the differential equation (7.30) and the boundary conditions (7.32) contain terms which are of even order in ξ and η ; hence, we assume, for Eq. (7.30), a variable separable solution of the type

$$w(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin m\pi\xi \sin n\pi\eta, \quad (7.33)$$

where m and n define the number of half waves that the plate buckles along the X - and Y -direction, respectively. With this representation for $w(\xi, \eta)$, the boundary conditions are identically satisfied. Substituting Eq. (7.33) in Eq. (7.30), we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \left\{ \left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) m^4 \pi^4 + 2p^2 m^2 n^2 \pi^4 + p^4 n^4 \pi^4 - \lambda^2 m^2 \pi^2 \right\} \sin m\pi\xi \sin n\pi\eta = 0. \quad (7.34)$$

The terms within the curly brackets can be zero only if the coefficient of each one of the terms is zero. Thus,

$$A_{mn} \left\{ \left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) m^4 \pi^4 + 2p^2 m^2 n^2 \pi^4 + p^4 n^4 \pi^4 - \lambda^2 m^2 \pi^2 \right\} = 0. \quad (7.35)$$

Since $A_{mn} \neq 0$, for Eq. (7.35) to be satisfied,

$$\lambda^2 = \frac{N_x}{D} a^2 = \left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) m^2 \pi^2 + 2p^2 n^2 \pi^2 + p^4 \frac{n^4}{m^2} \pi^2. \quad (7.36)$$

Hence,

$$N_x = \frac{\pi^2 D}{b^2} \left[\left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) \left(\frac{b}{a}\right)^2 + 2n^2 + n^4 \left(\frac{a}{mb}\right)^2 \right]. \quad (7.37)$$

Since we are interested in finding the lowest value of N_x at which the plate buckles, n must be equal to one. This implies that, irrespective of the plate aspect ratio, the plate buckles with one half sine wave along the Y -direction. The number of half waves along the X -direction that correspond to a minimum value of N_x can be found by taking the derivative of Eq. (7.37) with n equal to 1 and equating the resulting expression to zero. Thus,

$$\left(\frac{p}{m}\right)^2 = \left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right)^{1/2}. \quad (7.38)$$

Equation (7.38) indicates that, in inelastic buckling, the wave number along the X -direction is a function of the plate material property and the plate does not buckle into square plates as in elastic buckling [see Eq. (5.54)]. Therefore,

$$(N_x)_{cr} = \frac{\pi^2 D}{b^2} [2 + 2\left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right)^{1/2}], \quad (7.39)$$

$$(\sigma_x)_{cr} = \frac{\pi^2 E_s}{9} \left(\frac{t}{b}\right)^2 [2 + 2\left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right)^{1/2}]. \quad (7.40)$$

Comparing Eq. (7.40) with Eq. (5.59), we see that

$$(\sigma_x)_{cr} = \frac{4\pi^2 E}{12(1 - \nu_e^2)} \left(\frac{t}{b}\right)^2. \quad (7.41)$$

Defining η_p , the plasticity reduction factor, as

$$\eta_p = \frac{(\sigma_x)_{cr} \text{ (plastic)}}{(\sigma_x)_{cr} \text{ (elastic)}} \quad (7.42)$$

and substituting Eqs. (7.40) and (7.41) in Eq. (7.42), we obtain

$$\eta_p = \frac{(1 - \nu_e^2)}{(1 - \nu_p^2)} \frac{E_s}{E} \left[\frac{1}{2} + \frac{1}{2} \left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right)^{1/2} \right], \quad (7.43)$$

where ν_e and ν_p are the elastic and plastic values of Poisson's ratio, respectively.

The differential equation and the boundary conditions Stowell [1] derived for plastic buckling are applicable for a plate with simply-supported, clamped-free, and elastically restrained edges. Thus, as already noted, we need to know the values of two moduli E_s and E_t when following Stowell's approach. Further, since E_s depends on the stress level, a knowledge of the stress-strain curve of the plate material is also required.

7.3 BLEICH'S APPROACH

Consider a uniform rectangular plate as shown in Fig. 7.1. If it is subjected to a uniaxial compressive loading along the X -axis (as in the figure), then the stress σ_x is likely to exceed the proportional limit. Hence, the modulus governing the plate behaviour along the X -axis will be E_t . Also, the stress along the Y -axis will be the one due to Poisson's effect since the applied stress (load) along the Y -axis is zero. Thus, there exist two moduli when the critical load exceeds the elastic limit.

In Chapter 5, we derived the governing equation, i.e., (5.27), using the moment-curvature relations which assume that the plate behaviour is governed by only the elastic modulus. For the situation we are considering here, the moment-curvature relations take the form

$$M_x = -D(\tau \frac{\partial^2 w}{\partial x^2} + \nu \sqrt{\tau} \frac{\partial^2 w}{\partial y^2}), \quad (7.44a)$$

$$M_y = -D(\nu \sqrt{\tau} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}), \quad (7.44b)$$

$$M_{xy} = -M_y = D\sqrt{\tau}(1 - \nu) \frac{\partial^2 w}{\partial x \partial y}, \quad (7.44c)$$

where

$$\tau = E_t/E, \quad D = E_t^3/[12(1 - \nu^2)],$$

because the modulus along the X -direction is E_t and that along the Y -direction is E . In Eq. (7.44a), the second term within the parentheses is the contribution to M_x from the curvature along Y . Further, the value of $\sqrt{\tau}$ is taken as a mean of 0 and 1; alternatively, its value can be arbitrarily chosen.

The transverse shear forces along the X - and Y -axis become

$$Q_x = -D \frac{\partial}{\partial x} (\tau \frac{\partial^2 w}{\partial x^2} + \sqrt{\tau} \frac{\partial^2 w}{\partial y^2}), \quad (7.45a)$$

$$Q_y = -D \frac{\partial}{\partial y} (\sqrt{\tau} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}). \quad (7.45b)$$

Substituting Eqs. (7.44) in Eq. (5.19) and simplifying, we get

$$D(\tau \frac{\partial^4 w}{\partial x^4} + 2\sqrt{\tau} \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) + N_x \frac{\partial^2 w}{\partial x^2} = 0, \quad (7.46)$$

where N_x , N_y , and N_{xy} are the applied compressive loads. Equation (7.46) is valid only for

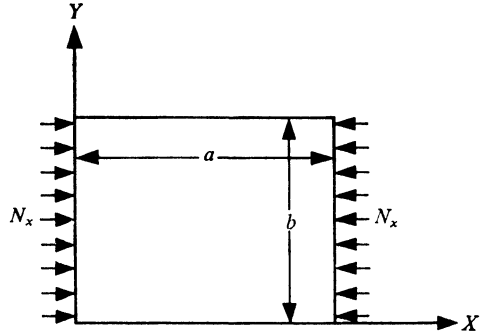


Fig. 7.1 Rectangular Plate under Uniaxial Compression.

a uniform plate compressed axially along the X -direction. If a plate is compressed only along the Y -direction, then the governing equation (7.46), moment-curvature relations (7.44), and transverse shear force equations (7.45) become

$$D\left(\frac{\partial^4 w}{\partial y^4} + 2\sqrt{\tau}\frac{\partial^4 w}{\partial x^2 \partial y^2} + \tau\frac{\partial^4 w}{\partial y^4}\right) + N_y\frac{\partial^2 w}{\partial y^2} = 0, \quad (7.47)$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\sqrt{\tau}\frac{\partial^2 w}{\partial y^2}\right), \quad (7.48a)$$

$$M_y = -D\left(\nu\sqrt{\tau}\frac{\partial^2 w}{\partial x^2} + \tau\frac{\partial^2 w}{\partial y^2}\right), \quad (7.48b)$$

$$M_{xy} = -M_{yx} = D\sqrt{\tau}(1 - \nu)\frac{\partial^2 w}{\partial x \partial y}, \quad (7.48c)$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \sqrt{\tau}\frac{\partial^2 w}{\partial y^2}\right), \quad (7.49a)$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\sqrt{\tau}\frac{\partial^2 w}{\partial x^2} + \tau\frac{\partial^2 w}{\partial y^2}\right). \quad (7.49b)$$

If a plate is acted upon by loads along the X - and Y -direction, Eqs. (7.47), (7.48), and (7.49) get further simplified and reduce to the equations derived in Section 5.2, with D replacing $D\tau$.

We shall demonstrate the application of the equations we have derived through numerical examples.

EXAMPLE 7.2 (Buckling load of rectangular plate in inelastic region) Consider a rectangular plate (Fig. 5.6) subjected to a compressive force N_x per unit length along the edges $x = 0, a$. In terms of the nondimensional parameters ξ and η , defined as

$$x = a\xi, \quad y = b\eta, \quad (7.50)$$

the governing equation (7.46) becomes

$$\tau\frac{\partial^4 w}{\partial \xi^4} + 2p^2\sqrt{\tau}\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4\frac{\partial^4 w}{\partial \eta^4} + \lambda^2\frac{\partial^2 w}{\partial \xi^2} = 0, \quad (7.51)$$

where

$$p = a/b, \quad \lambda^2 = N_x a^2/D.$$

For all edges simply-supported, the boundary conditions in the nondimensional form reduce to

$$w = D\tau\frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\xi = 0, 1), \quad (7.52a)$$

$$w = D\frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0, 1). \quad (7.52b)$$

A solution of the form

$$w(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin m\pi\xi \sin n\pi\eta \quad (7.53)$$

satisfies the boundary conditions (7.52) exactly. Substituting Eq. (7.53) in Eq. (7.51), we get

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} (\tau m^4 \pi^4 + 2p^2 \sqrt{\tau} m^2 n^2 \pi^4 + p^4 n^4 \pi^4 - \lambda^2 m^2 \pi^2) = 0. \quad (7.54)$$

For Eq. (7.54) to be satisfied, either A_{mn} is zero or the quantity within the parentheses is zero. The first alternative leads to a trivial solution, where, for all loads, the plate remains flat. For a nontrivial solution,

$$\lambda^2 = \frac{N_x}{D} a^2 = \tau m^2 \pi^2 + 2p^2 \sqrt{\tau} n^2 \pi^2 + p^4 \pi^2 \frac{n^4}{m^2}. \quad (7.55)$$

Therefore,

$$\begin{aligned} (N_x)_{cr} &= \frac{D\pi^2}{a^2} (\tau m^2 + 2p^2 \sqrt{\tau} n^2 + p^4 \frac{n^4}{m^2}) \\ &= \frac{D\pi^2}{b^2} \left(\frac{\sqrt{\tau} m}{p} + \frac{n^2}{m} p \right)^2. \end{aligned} \quad (7.56)$$

We are interested in finding that value of m and n which makes $(N_x)_{cr}$ a minimum. It is easy to visualize that as n increases, $(N_x)_{cr}$ also increases; hence, for a lowest value of $(N_x)_{cr}$, n must be equal to one. This implies that the plate buckles with one half sine wave along the Y -direction. The number of half waves along the X -direction that correspond to a minimum value of $(N_x)_{cr}$ can be found by taking the derivative of Eq. (7.56) with respect to m and equating the resulting expression to zero. Thus,

$$\frac{d(N_x)_{cr}}{dm} = 2D \frac{\pi^2}{b^2} \left(\sqrt{\tau} \frac{m}{p} + \frac{p}{m} \right) \left(\frac{\sqrt{\tau}}{p} - \frac{p}{m^2} \right) = 0. \quad (7.57)$$

From this, we get

$$m = p/(\tau)^{1/4}. \quad (7.58)$$

Substituting this value of m in Eq. (7.56), keeping n equal to one, we get

$$(N_x)_{cr} = 4D \frac{\pi^2}{b^2} \sqrt{\tau}. \quad (7.59)$$

Thus, for a simply-supported plate, the critical load depends on the elastic as well as the tangent modulus of the material. Further, from Eq. (7.58), we see that the wavelength along the X -direction depends on the material property and is larger than that for the corresponding elastic case [see Eq. (5.54)].

In general, the load $(N_x)_{cr}$ can be written as

$$(N_x)_{cr} = k_1 D \frac{\pi^2}{b^2}, \quad (7.60)$$

where

$$k_1 = (\sqrt{\tau} \frac{m}{p} + \frac{n^2}{m} p)^2. \quad (7.61)$$

To obtain the transition point at which the number of half waves m changes to $(m + 1)$ half waves, we equate the buckling coefficient k_1 for the two values of m since the other quantities are constants. That is,

$$\sqrt{\tau} \frac{m}{p} + \frac{p}{m} = \sqrt{\tau} \frac{(m + 1)}{p} + \frac{p}{(m + 1)}. \quad (7.62)$$

Simplifying, we get

$$p = \sqrt{m(m + 1)}(\tau)^{1/4}. \quad (7.63)$$

Thus, the aspect ratio at which the transition takes place also depends on the material property.

EXAMPLE 7.3 (Buckling load of rectangular plate in inelastic region) Consider a rectangular plate (Fig. 5.6) with the loading edges $\xi = 0, 1$ simply-supported and the edges $\eta = 0, 1$ clamped.

(i) *Stowell's approach* The differential equation governing the buckling behaviour is (7.30), i.e.,

$$(\frac{1}{4} + \frac{3}{4} \frac{E_1}{E_s}) \frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \bar{\lambda}^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \quad (7.30)$$

The boundary conditions along the edges $\xi = 0, 1$ are

$$w = (\frac{1}{4} + \frac{3}{4} \frac{E_1}{E_s}) \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{2} p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (7.64a)$$

and those along the edges $\eta = 0, 1$ are

$$w = \frac{\partial w}{\partial \eta} = 0. \quad (7.64b)$$

We choose a solution of the form

$$w(\xi, \eta) = F(\eta) \sin m\pi\xi. \quad (7.65)$$

This satisfies the boundary conditions along $\xi = 0, 1$. Substituting the expression for $w(\xi, \eta)$ in Eq. (7.30), we get an ordinary differential equation in $F(\eta)$, given as

$$[\frac{d^4 F}{d\eta^4} - \frac{2}{p^2} m^2 \pi^2 \frac{d^2 F}{d\eta^2} + \delta \frac{m^4 \pi^4}{p^4} F - \bar{\lambda}^2 \frac{m^2 \pi^2}{p^4} F] \sin m\pi\xi = 0, \quad (7.66)$$

where

$$\delta = \frac{1}{4} + \frac{3}{4} \frac{E_1}{E_s}.$$

Since $\sin m\pi\xi$ is not zero, the quantity within the square brackets must be zero. The general solution of Eq. (7.66) is

$$F(\eta) = A_1 \cosh \alpha\eta + A_2 \sinh \alpha\eta + A_3 \cos \beta\eta + A_4 \sin \beta\eta, \tag{7.67}$$

where

$$\begin{aligned} \alpha &= \left[\frac{m^2\pi^2}{p^2} + \left\{ (1 - \delta) \frac{m^4\pi^4}{p^4} + \bar{\lambda}^2 \frac{m^2\pi^2}{p^4} \right\}^{1/2} \right]^{1/2}, \\ \beta &= \left[- \frac{m^2\pi^2}{p^2} + \left\{ (1 - \delta) \frac{m^4\pi^4}{p^4} + \bar{\lambda}^2 \frac{m^2\pi^2}{p^4} \right\}^{1/2} \right]^{1/2}. \end{aligned} \tag{7.68}$$

The constants A_1, A_2, A_3 , and A_4 have to be determined from the edge conditions (7.64b). Following the same steps as in Example 5.2, we can obtain the critical load N_x from the transcendental equation

$$2(1 - \cosh \alpha \cos \beta) = \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta} \right) \sinh \alpha \sin \beta, \tag{7.69}$$

where α and β are as defined by Eqs. (7.68).

Table 7.1 shows the variation of critical stress with the aspect ratio for an aluminium alloy plate.

Table 7.1 Values of $\bar{\lambda}_{cr}$ for Axially Compressed Plate, Loading Edges Simply-Supported and Other Two Edges Clamped

Given

$$E = 0.588 \times 10^9 \text{ N/m}^2, \quad t/b = 0.04, \quad E_t/E_s = 0.9, \quad \tau = 0.9.$$

Aspect ratio	0.5	0.8	1.0	1.2	1.5	2.0	2.5
$\bar{\lambda}_{cr}$ (Stowell)	4.2705	6.7375	8.5410	9.8645	12.4523	16.3879	20.4911
$\bar{\lambda}_{cr}$ (Bleich)	4.2063	6.6605	8.4127	9.7256	12.3014	16.1707	20.2086

$$\bar{D} = E_s t^3/9, \quad \bar{\lambda} = N_x(a^2/\bar{D}).$$

(ii) *Bleich's approach* The differential equation governing the behaviour of the plate is

$$\tau \frac{\partial^4 w}{\partial x^4} + 2\sqrt{\tau} \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2} = 0. \tag{7.46}$$

In terms of the nondimensional parameters ξ and η , Eq. (7.46) can be written as

$$\tau \frac{\partial^4 w}{\partial \xi^4} + 2\sqrt{\tau} p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \lambda^2 \frac{\partial^2 w}{\partial \xi^2} = 0, \quad (7.70)$$

where

$$\lambda^2 = \frac{N_x a^2}{D}.$$

The boundary conditions along the edges $\xi = 0, 1$ are given by (7.52a). The differential equation (7.70) and the boundary conditions (7.52a) are satisfied by the expression

$$w(\xi, \eta) = F(\eta) \sin m\pi\xi \quad (m = 1, 2, 3). \quad (7.71)$$

Substituting Eq. (7.71) in Eq. (7.70), we obtain the fourth order ordinary differential equation

$$\left[\frac{d^4 F}{d\eta^4} - 2\sqrt{\tau} \frac{m^2 \pi^2}{p^2} \frac{d^2 F}{d\eta^2} + \left(\frac{\tau}{p^4} m^4 \pi^4 - \frac{\lambda^2}{p^4} m^2 \pi^2 \right) F \right] \sin m\pi\xi = 0. \quad (7.72)$$

Since $\sin m\pi\xi$ is not zero, the quantity within the square brackets has to be zero. This leads to

$$\frac{d^4 F}{d\eta^4} - 2\sqrt{\tau} \frac{m^2 \pi^2}{p^2} \frac{d^2 F}{d\eta^2} + \left(\frac{\tau}{p^4} m^4 \pi^4 - \frac{\lambda^2}{p^4} m^2 \pi^2 \right) F = 0. \quad (7.73)$$

The solution of this equation is

$$F(\eta) = A_1 \cosh \alpha\eta + A_2 \sinh \alpha\eta + A_3 \cos \beta\eta + A_4 \sin \beta\eta, \quad (7.74)$$

where

$$\begin{aligned} \alpha &= \left(\sqrt{\tau} \frac{m^2 \pi^2}{p^2} + \frac{\lambda}{p^2} m\pi \right)^{1/2}, \\ \beta &= \left(-\sqrt{\tau} \frac{m^2 \pi^2}{p^2} + \frac{\lambda}{p^2} m\pi \right)^{1/2}. \end{aligned} \quad (7.75)$$

The constants A_1, A_2, A_3 , and A_4 have to be determined from the edge conditions (7.64b). Following the same steps as in Example 5.2, we obtain the transcendental equation

$$2(1 - \cosh \alpha \cos \beta) = \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta} \right) \sinh \alpha \sin \beta. \quad (7.76)$$

From the roots of this equation, we get α and β . The buckling parameter λ can now be obtained.

Table 7.1 shows the variation of critical stress with the aspect ratio for an aluminium alloy plate.

EXAMPLE 7.4 (Buckling of rectangular plate) Consider a rectangular plate along with its edge view (Fig. 7.2). Assume that the two restraining plates are symmetric with no loads

acting on them. Further, suppose that these plates are stiff enough so that they do not

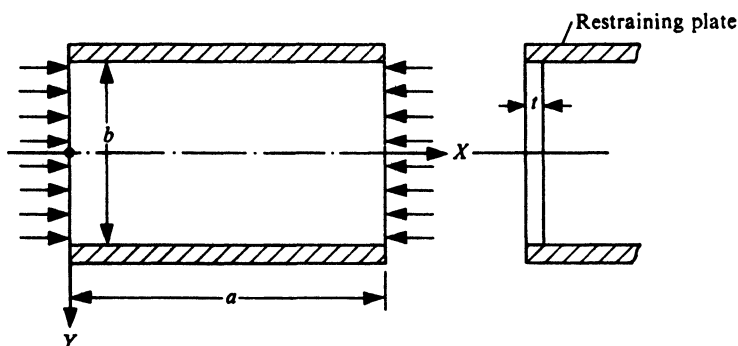


Fig. 7.2 Rectangular Plate with Its Edge View.

allow any transverse displacement of the plate along their edges. Since the active plate and the restraining plates are rigidly connected along the edges, the active plate and the restraining plates rotate by an amount such that the angle between them is maintained at 90° .

Since the structure and the loading are symmetric, we have accordingly chosen the coordinate system (see Fig. 7.2).

(i) *Bleich's approach* In terms of the nondimensional parameters ξ and η , defined as

$$x = a\xi, \quad y = \frac{b}{2}\eta, \quad (7.77)$$

the differential equation (7.30) can be written as

$$\tau \frac{\partial^4 w}{\partial \xi^4} + \sqrt{\tau} p_1^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p_1^4 \frac{\partial^4 w}{\partial \eta^4} + \lambda^2 \frac{\partial^2 w}{\partial \xi^2} = 0, \quad (7.78)$$

where

$$\lambda^2 = \frac{N_x a^2}{D}, \quad p_1 = \frac{2a}{b} = 2p.$$

Even though we have replaced p by p_1 , the analysis procedure here remains the same as in Example 7.3. The ordinary differential equation governing $F(\eta)$, i.e., (7.66), then becomes

$$\frac{d^4 F}{d\eta^4} - \sqrt{\tau} \frac{m^2 \pi^2}{p_1^2} \frac{d^2 F}{d\eta^2} + \left(\frac{\tau}{p_1^4} m^4 \pi^4 - \frac{\lambda^2}{p_1^2} m^2 \pi^2 \right) F = 0. \quad (7.79)$$

In view of the symmetry of the structure, the solution of Eq. (7.79) can be written as

$$F(\eta) = A_1 \cosh \alpha \eta + A_3 \cos \beta \eta, \quad (7.80)$$

where

$$\alpha = (\sqrt{\tau} \frac{m^2 \pi^2}{p_1^2} + \frac{\lambda}{p_1^2} m \pi)^{1/2}, \quad (7.81)$$

$$\beta = (-\sqrt{\tau} \frac{m^2 \pi^2}{p_1^2} + \frac{\lambda}{p_1^2} m \pi)^{1/2}.$$

The constants A_1 and A_3 have to be determined from the boundary conditions along the edges $\eta = \pm 1$. These conditions are

$$F(\eta) = 0, \quad (7.82a)$$

$$\phi = \bar{\phi}. \quad (7.82b)$$

To incorporate these boundary conditions, we need to express ϕ and $\bar{\phi}$ in terms of the deflection $w(\xi, \eta)$. As the active plate deflects, a moment M_y develops at the edges, causing a rotation of the active, and therefore of the restraining, plate. The bending moment M_y is assumed to be proportional to $\bar{\phi}$ (small deformation theory assumption). This is expressed as

$$M_y = -\bar{\zeta} \bar{\phi}, \quad (7.83)$$

where $\bar{\zeta}$ depends on the dimensions of the restraining plate. (Bleich [2] has derived the expression for $\bar{\zeta}$ for various dimensions of the restraining plate.) The moment M_y is related to the deflection w , according to Eq. (7.44b), as

$$M_y = -D(\nu \sqrt{\tau} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})_{y=\pm b/2}. \quad (7.84)$$

As the edge is supported everywhere along the X -direction, $\partial^2 w / \partial x^2 = 0$. Equation (7.84) therefore reduces to

$$M_y = -D(\frac{\partial^2 w}{\partial y^2})_{y=\pm b/2}. \quad (7.85)$$

Substituting for M_y in Eq. (7.84), we get

$$\bar{\phi} = \frac{D}{\bar{\zeta}} (\frac{\partial^2 w}{\partial y^2})_{y=\pm b/2}. \quad (7.86)$$

According to Eq. (7.82b), $\phi = \bar{\phi}$ and, since $\phi = \pm (\partial w / \partial y)_{y=\pm b/2}$, Eq. (7.86) takes the form

$$(\frac{\partial w}{\partial y} \pm \frac{D}{\bar{\zeta}} \frac{\partial^2 w}{\partial y^2})_{y=\pm b/2} = 0. \quad (7.87)$$

In terms of the nondimensional parameters η and ζ_b , we find Eq. (7.87) reduces to

$$(\frac{dF}{d\eta} \pm \zeta_b \frac{d^2 F}{d\eta^2})_{\eta=\pm 1} = 0, \quad (7.88)$$

where

$$\zeta_b = \frac{2}{b} \frac{D}{\bar{\zeta}}.$$

It can be easily seen that when $\zeta_b = 0$, the plate edge is clamped, and when $\zeta_b = \infty$, the edge is hinged. Introducing Eq. (7.80) in the boundary conditions (7.82a) and (7.88), we get two homogeneous algebraic equations, namely,

$$A_1 \cosh \alpha + A_3 \cos \beta = 0, \quad (7.89a)$$

$$A_1 \alpha \sinh \alpha - A_3 \beta \sin \beta + \zeta_b (A_1 \alpha^2 \cosh \alpha - A_3 \beta^2 \cos \beta) = 0. \quad (7.89b)$$

For a nontrivial solution, the determinant of the coefficients of A_1 and A_3 must be zero. This leads to the stability equation

$$\alpha \tanh \alpha + \beta \tan \beta + \zeta_b (\alpha^2 + \beta^2) = 0. \quad (7.90)$$

The critical load N_x can be obtained by solving this transcendental equation.

(ii) *Stowell's approach* The equation governing the plate behaviour is (7.30) and, with p replaced by p_1 , this becomes

$$\left(\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}\right) \frac{\partial^4 w}{\partial \xi^4} + 2p_1^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p_1^4 \frac{\partial^4 w}{\partial \eta^4} + \bar{\lambda}^2 \frac{\partial^2 w}{\partial \xi^2} = 0, \quad (7.91)$$

where $p_1 = 2a/b$ and $\bar{\lambda}^2 = N_x a^2 / D$. A solution of the type

$$w(\xi, \eta) = F(\eta) \sin m\pi\xi \quad (7.92)$$

satisfies the boundary conditions along the edges $\xi = 0, 1$. Substituting Eq. (7.92) in Eq. (7.91), we obtain a fourth order ordinary differential equation in $F(\eta)$, namely,

$$\frac{d^4 F}{d\eta^4} - \frac{2m^2\pi^2}{p_1^2} \frac{d^2 F}{d\eta^2} + \frac{1}{p_1^4} (\delta m^4 \pi^4 - \bar{\lambda}^2 m^2 \pi^2) F = 0, \quad (7.93)$$

where

$$\delta = \frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}.$$

In view of the symmetry of the structure, the general solution of Eq. (7.93) is

$$F(\eta) = A_2 \cosh \alpha_1 \eta + A_4 \cos \beta_1 \eta, \quad (7.94)$$

where

$$\alpha_1 = \left[\frac{m^2 \pi^2}{p_1^2} + \left\{ (1 - \delta) \frac{m^4 \pi^4}{p_1^4} + \bar{\lambda}^2 \frac{m^2 \pi^2}{p_1^4} \right\}^{1/2} \right]^{1/2}, \quad (7.95a)$$

$$\beta_1 = \left[-\frac{m^2 \pi^2}{p_1^2} + \left\{ (1 - \delta) \frac{m^4 \pi^4}{p_1^4} + \bar{\lambda}^2 \frac{m^2 \pi^2}{p_1^4} \right\}^{1/2} \right]^{1/2}. \quad (7.95b)$$

The bending moment M_y is given by

$$M_y = -D \left(\frac{1}{2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right).$$

The constants A_2 and A_4 in Eq. (7.94) have to be determined from the boundary conditions

along the edges $\eta = \pm 1$. These conditions are

$$F(\eta) = 0, \quad (7.96a)$$

$$\phi = \bar{\phi}. \quad (7.96b)$$

The bending moment M_y is assumed to be proportional to $\bar{\phi}$. This is expressed as

$$M_y = -\bar{\zeta}\bar{\phi}. \quad (7.97)$$

The moment M_y is related to the deflection w as

$$M_y = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \quad (7.98)$$

As the edge is supported all along the X -direction, $\partial^2 w / \partial x^2 = 0$. Equation (7.98) therefore reduces to

$$M_y = -D \frac{\partial^2 w}{\partial y^2}. \quad (7.99)$$

Substituting for M_y in Eq. (7.99), we get

$$\bar{\phi} = \frac{D}{\bar{\zeta}} \left(\frac{\partial^2 w}{\partial y^2} \right)_{y=\pm b/2}. \quad (7.100)$$

Since $\phi = \pm(\partial w / \partial y)_{y=\pm b/2}$, Eq. (7.100) becomes

$$\left(\frac{\partial w}{\partial y} + \frac{D}{\bar{\zeta}} \frac{\partial^2 w}{\partial y^2} \right)_{y=\pm b/2} = 0. \quad (7.101)$$

In terms of the nondimensional parameters η and ζ_b and making use of Eq. (7.96b), we find Eq. (7.101) reduces to

$$\left(\frac{dF}{d\eta} \pm \zeta_b \frac{d^2 F}{d\eta^2} \right)_{\eta=\pm 1} = 0, \quad (7.102)$$

where

$$\zeta_b = \frac{2}{b} \frac{D}{\bar{\zeta}}.$$

Introducing Eq. (7.94) in the boundary conditions (7.96a) and (7.102), we obtain two homogeneous algebraic equations, namely,

$$A_2 \cosh \alpha_1 + A_4 \cos \beta_1 = 0, \quad (7.103a)$$

$$A_2 \alpha_1 \sinh \alpha_1 - A_4 \beta_1 \sin \beta_1 + \zeta_b (A_2 \alpha_1^2 \cosh \alpha_1 - A_4 \beta_1^2 \cos \beta_1) = 0. \quad (7.103b)$$

For a nontrivial solution, the determinant of the coefficients of A_2 and A_4 must be zero. This leads to the stability equation

$$\alpha_1 \tanh \alpha_1 + \beta_1 \tan \beta_1 + \zeta_b (\alpha_1^2 + \beta_1^2) = 0. \quad (7.104)$$

The critical load N_x can now be obtained from this equation.

7.4 BIAXIAL LOADING

In Section 7.1, we noted that if a plate is subjected to inplane biaxial loading, then the

stresses in the plate along both the X - and Y -direction may exceed the elastic limit. For example, if a plate is subjected to pure shear loading, then the corresponding principal stresses are $\sigma_1 = -\sigma_2 = \tau_{xy}$. In such a case, a reduced modulus of elasticity is effective in both the directions, and therefore the buckling load in the inelastic region is obtained by replacing the elastic modulus by an appropriate reduced modulus. In what follows, we shall show how the Stowell and Bleich approaches (see Sections 7.2 and 7.3) can be applied to inelastic plates.

7.4.1 Bleich's Approach

Consider a rectangular plate subjected to a biaxial inplane loading. The governing equation (7.46) we derived is applicable only when the load is applied along the X -axis. The mixed derivative term in this equation results from the curvatures along the X - and Y -direction. Equation (7.46) therefore gets modified to

$$D_r \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad (7.105)$$

or

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D_r} \frac{\partial^2 w}{\partial x^2} + \frac{N_y}{D_r} \frac{\partial^2 w}{\partial y^2} = 0, \quad (7.106)$$

where D_r , the reduced modulus, is equal to D_T .

The moment and shear force equations for biaxial loading reduce to

$$M_x = -D_r \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \quad (7.107)$$

$$M_y = -D_r \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \quad (7.108)$$

$$M_{xy} = -M_{yx} = D_r - (1 - \nu) \frac{\partial^2 w}{\partial x \partial y}, \quad (7.109)$$

$$Q_x = -D_r \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right), \quad (7.110)$$

$$Q_y = -D_r \frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} \right). \quad (7.111)$$

The method of solution from here onwards remains the same as given in Section 7.3.

7.4.2 Stowell's Approach

The constitutive equations derived in Section 7.2.1 are also valid for biaxial loading. However, to derive the governing equation, the moment-curvature relations (7.14), (7.19), and (7.22) have to be modified. This is done as follows. As we know, up to the point of buckling, Eqs. (7.5) give the state of stress. Further, at buckling, the bending of the plate gives rise to a variation in the strains ϵ_x , ϵ_y , and γ_{xy} . In view of the small deflection assumption, we neglect the change in the strain due to the middle surface strain. Therefore, the variations

in the curvature caused by a change in the strain are

$$\delta\epsilon_x = -z\delta\chi_1, \quad \delta\epsilon_y = -z\delta\chi_2, \quad \delta\gamma_{xy} = -2z\delta\chi_3. \quad (7.112)$$

Thus, the change in the stress $\delta\sigma_x$ is

$$\begin{aligned} \delta\sigma_x &= \frac{4}{3}E_s(\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y) + \frac{4}{3}(\epsilon_x + \frac{1}{2}\epsilon_y)\delta E_s \\ &= \frac{4}{3}E_s(\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y) + \frac{4}{3}(\epsilon_x + \frac{1}{2}\epsilon_y)\delta\left(\frac{\bar{\sigma}}{\bar{\epsilon}}\right) \\ &= \frac{4}{3}E_s(\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y) - \frac{4}{3}(\epsilon_x + \frac{1}{2}\epsilon_y)\frac{1}{\bar{\epsilon}}\left(\frac{\bar{\sigma}}{\bar{\epsilon}} - \frac{d\bar{\sigma}}{d\bar{\epsilon}}\right)\delta\bar{\epsilon}. \end{aligned} \quad (7.113)$$

The work done by the internal forces when the plate passes from the undeflected to a deflected state is

$$\bar{\sigma}\delta\bar{\epsilon} = \sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + 2\tau_{xy}\delta\gamma_{xy}. \quad (7.114)$$

Substituting Eqs. (7.112) in Eq. (7.114), we have

$$\delta\bar{\epsilon} = -\frac{z}{\bar{\sigma}}(\sigma_x\delta\chi_1 + \sigma_y\delta\chi_2 + 2\tau_{xy}\delta\chi_3). \quad (7.115)$$

Substituting for $\delta\bar{\epsilon}$ in Eq. (7.113), we get

$$\delta\sigma_x = \frac{4}{3}E_s(\delta\epsilon_x + \frac{1}{2}\delta\epsilon_y) + \frac{4}{3}(\epsilon_x + \frac{1}{2}\epsilon_y)\frac{1}{\bar{\epsilon}}(E_s - E_t)\frac{z}{\bar{\sigma}}(\sigma_x\delta\chi_1 + \sigma_y\delta\chi_2 + 2\tau_{xy}\delta\chi_3) \quad (7.116)$$

$$= \frac{4}{3}E_s(-z\delta\chi_1 - \frac{1}{2}z\delta\chi_2) + \frac{\sigma_x}{E_s}\frac{z}{\bar{\sigma}}(E_s - E_t)(\sigma_x\delta\chi_1 + \sigma_y\delta\chi_2 + 2\tau_{xy}\delta\chi_3). \quad (7.117)$$

The change in the moment M_x is given as

$$\begin{aligned} \delta M_x &= \int_{-t/2}^{t/2} \delta\sigma_x z \, dz \\ &= \int_{-t/2}^{t/2} \left[\frac{4}{3}E_s(-z^2\delta\chi_1 - \frac{1}{2}z^2\delta\chi_2) + \frac{\sigma_x}{E_s}\frac{z^2}{\bar{\sigma}}(E_s - E_t)(\sigma_x\delta\chi_1 + \sigma_y\delta\chi_2 + 2\tau_{xy}\delta\chi_3) \right] dz \\ &= -\frac{4}{3}E_s\frac{t^3}{12}[(\delta\chi_1 + \frac{1}{2}\delta\chi_2) - \frac{3}{4}\frac{\sigma_x}{\bar{\sigma}E_s^2}(E_s - E_t)(\sigma_x\delta\chi_1 + \sigma_y\delta\chi_2 + 2\tau_{xy}\delta\chi_3)] \\ &= -D[(1 - \frac{3}{4}\lambda\frac{\sigma_x^2}{\bar{\sigma}^2})\delta\chi_1 + \frac{1}{2}(1 - \frac{3}{2}\lambda\frac{\sigma_x\sigma_y}{\bar{\sigma}^2})\delta\chi_2 - \frac{3}{2}\frac{\sigma_x\tau_{xy}}{\bar{\sigma}^2}\lambda\delta\chi_3], \end{aligned} \quad (7.118)$$

where

$$D = \frac{E_s t^3}{9}, \quad \lambda = 1 - \frac{E_t}{E_s}.$$

Similarly,

$$\delta M_y = -D[(1 - \frac{3}{8}\lambda\frac{\sigma_x^2}{\bar{\sigma}^2})\delta\chi_2 + \frac{1}{8}(1 - \frac{3}{8}\lambda\frac{\sigma_x\sigma_y}{\bar{\sigma}^2})\delta\chi_1 - \frac{3}{8}\lambda\frac{\sigma_y\tau_{xy}}{\bar{\sigma}^2}\delta\chi_3], \quad (7.119)$$

$$\delta M_{xy} = -\frac{D}{2}[(1 - \lambda\frac{3\tau_{xy}^2}{\bar{\sigma}^2})\delta\chi_3 - \frac{3}{8}\lambda(\frac{\sigma_x\tau_{xy}}{\bar{\sigma}^2}\delta\chi_1 + \frac{\sigma_y\tau_{xy}}{\bar{\sigma}^2}\delta\chi_2)]. \quad (7.120)$$

From Eq. (5.19), the equilibrium equation for a plate element is

$$\frac{\partial^2}{\partial x^2}(\delta M_x) + \frac{\partial^2}{\partial x^2}(\delta M_y) - 2\frac{\partial^2}{\partial x \partial y}(\delta M_{xy}) + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0. \quad (7.121)$$

Equation (7.24) gives the relation between the curvatures and the deflection w . Hence,

$$\frac{\partial^2}{\partial x^2}(\delta M_x) = -D[(1 - \frac{3}{8}\lambda\frac{\sigma_x^2}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial x^4} + \frac{1}{8}(1 - \frac{3}{8}\lambda\frac{\sigma_x\sigma_y}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{3}{8}\lambda\frac{\sigma_x\tau_{xy}}{\bar{\sigma}^2}\frac{\partial^4 w}{\partial x^3 \partial y}], \quad (7.122a)$$

$$\frac{\partial^2}{\partial y^2}(\delta M_y) = -D[(1 - \frac{3}{8}\lambda\frac{\sigma_y^2}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial y^4} + \frac{1}{8}(1 - \frac{3}{8}\lambda\frac{\sigma_x\sigma_y}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{3}{8}\lambda\frac{\sigma_y\tau_{xy}}{\bar{\sigma}^2}\frac{\partial^4 w}{\partial x \partial y^3}], \quad (7.122b)$$

$$\frac{\partial^2}{\partial x \partial y}(\delta M_{xy}) = -\frac{D}{2}[(1 - \lambda\frac{3\tau_{xy}^2}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{3}{8}\lambda(\frac{\sigma_x\tau_{xy}}{\bar{\sigma}^2}\frac{\partial^4 w}{\partial x^3 \partial y} + \frac{\sigma_y\tau_{xy}}{\bar{\sigma}^2}\frac{\partial^4 w}{\partial x \partial y^3})]. \quad (7.122c)$$

Substituting Eqs. (7.122) in Eq. (7.121) with $N_{xy} = 0$, we get

$$(1 - \frac{3}{8}\lambda\frac{\sigma_x^2}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial x^4} + 2(1 - \frac{3}{8}\lambda\frac{\sigma_x\sigma_y}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial x^2 \partial y^2} + (1 - \frac{3}{8}\lambda\frac{\sigma_y^2}{\bar{\sigma}^2})\frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} + \frac{N_y}{D}\frac{\partial^2 w}{\partial y^2} = 0, \quad (7.123)$$

where $\bar{\sigma} = (\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y)^{1/2}$.

As can be observed, Stowell's approach is more involved than Bleich's approach.

7.5 NUMERICAL TECHNIQUES

The two numerical techniques, namely, the finite difference and finite element techniques (see Section 5.8), are both applicable to inelastic buckling. When the former method is applied, the differential equations (7.30) and (7.51) have to be written in a finite difference form, using the differences derived in Section 5.8.1. For a given problem, τ or E_t/E_s is known *a priori*, and hence the method basically reduces to solving the differential equation for an elastic plate with different coefficients. The application of the finite element technique, on the other hand, is comparatively more involved, and therefore we do not discuss it here.

PROBLEMS

7.1 Determine the critical load for a simply-supported square plate, axially compressed (see Fig. 7.3). The thickness of the plate is such that the stresses within it exceed the elastic limit before buckling occurs. Assume $\tau = 0.9$.

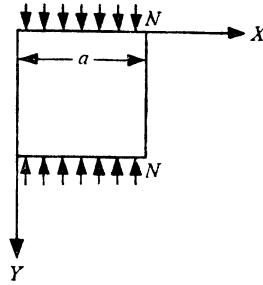


Fig. 7.3 Problem 7.1.

7.2 Solve Problem 7.1 by using Stowell's approach. Given $E_t/E_s = 0.85$.

7.3 Determine the critical load of a simply-supported square plate axially compressed in the X -direction by a linearly varying load, as shown in Fig. 7.4. (Hint: Use the Galerkin technique.)

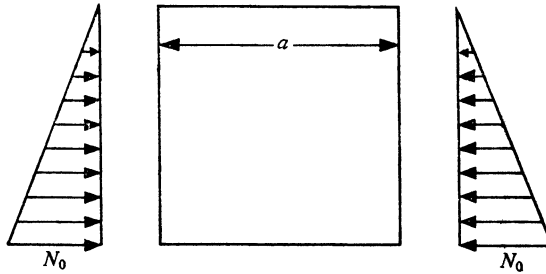


Fig. 7.4 Problem 7.3.

7.4 The web plate of the T -section (see Fig. 7.5) is subjected to a uniform axially compressive load. The loading edges are assumed to be simply-supported. The coefficient of

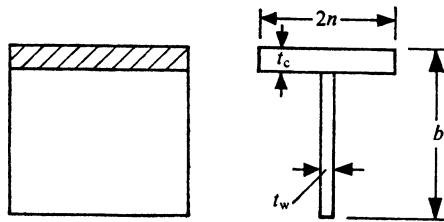


Fig. 7.5 Problem 7.4.

restraint ζ_b is given by

$$\zeta_b = \frac{t_w^3}{t_c^3} \frac{1}{1 - (0.016) \frac{t_w^2 h^2}{t_c^2 b^2}}$$

where $h = 0.025$ m, $b = 0.1$ m, $t_w = 0.001$ m, and $t_c = 0.003$ m. Determine the critical load N_x .

7.5 A 0.005-m-thick aluminium plate of length $a = 0.8$ m and width $b = 0.6$ m is simply-supported on all edges. It is subjected to uniform compressive loads in both the directions. If $N_y = 0.6N_x$, determine the value of N_x for inelastic buckling. Let $E = 58.86$ GPa, $\nu = 0.9$, and $\tau = 0.3$.

7.6 Determine the critical stress for a web plate of a girder, subjected to a pure shear stress τ_{xy} .

7.7 Derive an expression for the critical load in the inelastic region of a plate, simply-supported along the loaded edges and elastically restrained along the other two edges. The restraints are due to rotational springs of stiffnesses β_1 and β_2 per unit length per unit rotation and linear springs of stiffnesses k_1 and k_2 per unit length per unit deflection.

7.8 Plot the interaction curve for buckling in the inelastic region for a square plate, simply-supported on all sides and subjected to biaxial inplane compressive stresses.

7.9 Use Stowell's approach to derive the governing equation for inelastic buckling when only σ_x and τ_{xy} are acting in the plane of a plate.

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8

Postbuckling Behaviour of Plates

8.1 INTRODUCTION

In Chapter 1, we discussed the buckling of columns. Experience has shown that a column fails when the critical load is reached. Experimentally too it has been shown that the strength of a column tends to zero at the critical load and a slight increase thereafter in the applied load is sufficient for the column to collapse.

In Chapter 5, we showed that the behaviour of a plate is entirely different from that of a column because of the two-dimensional nature of the former. A plate starts to buckle when the critical load is reached. It is usually able to resist additional load thereafter and may not fail till the load applied is considerably higher than the critical load. This is so because a plate is supported on all sides. Such behaviour of a plate is of much consequence when the structural weight has to be minimized to safely carry a given load. For example, in an aircraft, the wing spar is enclosed within the wing surface. The spar is made up of flanges with a thin web. As the initial buckling load is reached, the web buckles, but then since the flanges support the web, the web can carry additional load. It has been observed that this excess load can be 20–30 times the initial buckling load. In structural engineering and when designing ships, the increase in the actual load carrying capacity is less impressive because b/t is quite large and is generally of the order of 2. Even in a stiffened structure, for instance, in the sheet-stringer panels of an aircraft, the ultimate load could be several times the buckling load.

In Chapters 5 and 6, we derived the equilibrium equation for the buckling of elastic isotropic, stiffened, and composite plates, assuming the deformations to be small. To take into account the large deformations which occur in the postbuckling range, these equations need to be modified. In this chapter, we shall do this and apply the governing equation so obtained to some typical problems.

8.2 GOVERNING EQUATION

We shall derive here the governing equation for the postbuckling (finite deformation) of a thin plate. In doing this, we shall assume that the plate is subjected to all possible inplane stresses, namely, σ_x , σ_y , and τ_{xy} . Also, we shall treat as valid assumptions (i) and (ii) made in Section 5.2, except that the membrane stresses that develop due to the stretching of the midplane would no longer be negligible and would need to be incorporated in our analysis.

As has already been noted, a plate element is acted upon by the stress resultants N_x , N_y ,

N_{xy} , and M_x , M_y , M_{xy} . Of these, the inplane quantities, namely, N_x , N_y , and N_{xy} , change because of the finite deformation; the moments, however, remain unchanged. Hence, the moment equilibrium equations (5.14) and (5.16) are still valid.

Figure 8.1 shows the inplane forces acting on a differential element; in it, we have retained only the first order terms. As depicted, in addition to the change in the slope, the internal

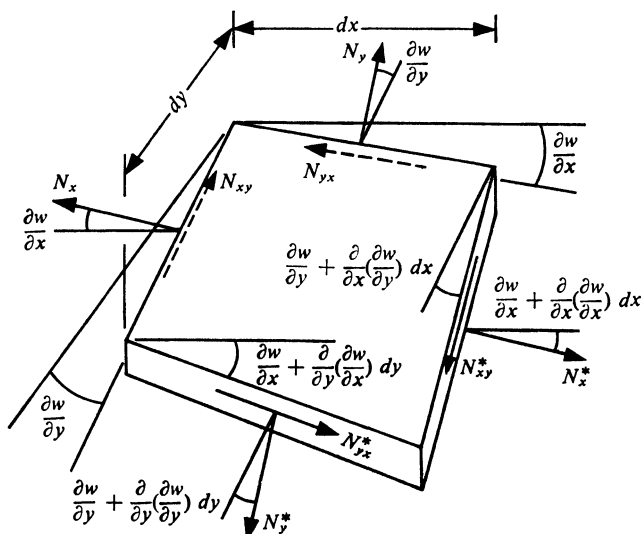


Fig. 8.1 Inplane Forces Acting on Plate Element.

inplane stress changes when we move from one side of the element to the other. This change in the internal inplane stress results when the small deformation assumption is relaxed. In Fig. 8.1,

$$N_x^* = N_x + \frac{\partial}{\partial x} N_x dx, \quad N_y^* = N_y + \frac{\partial}{\partial y} N_y dy,$$

$$N_{xy}^* = N_{xy} + \frac{\partial}{\partial x} N_{xy} dx, \quad N_{yx}^* = N_{yx} + \frac{\partial}{\partial y} N_{yx} dy,$$

$$\frac{\partial w^*}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) dx, \quad \frac{\partial w^*}{\partial y} = \frac{\partial w}{\partial y} + \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} \right) dy.$$

The projections of the inplane forces in the X - and Y -direction are

$$(N_x^* \frac{\partial w^*}{\partial x} - N_x \frac{\partial w}{\partial x}) dy + [N_{yx}^* (\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial v \partial x} dx) - N_{yx} \frac{\partial w}{\partial x}] dx = 0, \quad (8.1a)$$

$$(N_y^* \frac{\partial w^*}{\partial y} - N_y \frac{\partial w}{\partial y}) dx + [N_{xy}^* (\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y \partial x} dx) - N_{xy} \frac{\partial w}{\partial y}] dy = 0. \quad (8.1b)$$

These, on simplification, yield

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad (8.2a)$$

$$\frac{\partial N_x}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0. \quad (8.2b)$$

Since the inplane forces are inclined to the sides, there exists a transverse component of these forces along the Z -direction. This is obtained by taking the z -component of all the forces. The z -component of the force N_x is

$$-N_x \frac{\partial w}{\partial y} dy + N_x^* \frac{\partial w^*}{\partial y} dy.$$

Substituting for the quantities with an asterisk and neglecting terms of higher order, we get

$$N_x \frac{\partial^2 w}{\partial x^2} dx dy + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dx dy. \quad (8.3a)$$

Proceeding in a similar manner, we find the z -component of each of the forces N_y , N_{xy} , and N_{yx} ($=N_{xy}$) along the Z -direction is

$$N_y \frac{\partial^2 w}{\partial y^2} dx dy + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} dx dy, \quad (8.3b)$$

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dx dy, \quad (8.3c)$$

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy + \frac{\partial N_{xy}}{\partial y} \frac{\partial w}{\partial x} dx dy. \quad (8.3d)$$

The net vertical component of the inplane forces is

$$(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}) dx dy. \quad (8.4)$$

Here, in view of Eqs. (8.2), the rest of the terms of Eqs. (8.3) are identically zero. Combining Eq. (8.4) with Eq. (5.27), we find the nonlinear equilibrium equation along the Z -direction is

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0, \quad (8.5)$$

where the coefficients N_x , N_y , and N_{xy} are not applied quantities as in Eq. (5.27). Instead, N_x , N_y , N_{xy} here are the inplane forces generated as a consequence of the large deformation of the plate, and hence they vary from point to point within the domain of the plate. Equation (8.5) is a function of four variables, namely, w , N_x , N_y , and N_{xy} . In turn, each of these variables is a function of x and y . In order to solve this equation, we need additional equations involving these variables. As the deflection is large, the fibres, in addition to bending, stretch, and hence the strain-displacement relations (5.22) need to be modified to take

stretching into account. Defining u_0 and v_0 as the displacements of a point in the middle plane, the normal and shear strains in the middle plane are

$$\epsilon_x = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad (8.6a)$$

$$\epsilon_y = \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \quad (8.6b)$$

$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}. \quad (8.6c)$$

Eliminating u_0 and v_0 , we get

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}. \quad (8.7)$$

The strain terms in Eq. (8.7) can be eliminated by using the stress-strain relations. These relations can be written in terms of force per unit length as

$$N_x = \frac{Et}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y), \quad (8.8a)$$

$$N_y = \frac{Et}{1 - \nu^2} (\epsilon_y + \nu \epsilon_x), \quad (8.8b)$$

$$N_{xy} = \frac{Et}{2(1 + \nu)} \gamma_{xy}. \quad (8.8c)$$

Using Eqs. (8.8) and (8.2) in Eq. (8.7), we get

$$\frac{1}{Et} \left(\frac{\partial^2 N_x}{\partial y^2} + \frac{\partial^2 N_y}{\partial x^2} - 2 \frac{\partial^2 N_{xy}}{\partial x \partial y} \right) = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}. \quad (8.9)$$

In terms of the force function $\phi(x, y)$, defined as

$$N_x = t \frac{\partial^2 \phi}{\partial y^2}, \quad (8.10a)$$

$$N_y = t \frac{\partial^2 \phi}{\partial x^2}, \quad (8.10b)$$

$$N_{xy} = -t \frac{\partial^2 \phi}{\partial x \partial y}, \quad (8.10c)$$

Eq. (8.9) reduces to

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right]. \quad (8.11)$$

In view of Eqs. (8.10), Eq. (8.5) takes the form

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{t}{D} \left(\frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right). \quad (8.12)$$

Thus, the governing equations defining the postbuckling behaviour of an isotropic plate reduce to two simultaneous equations, namely, (8.11) and (8.12), in terms of ϕ and w . These equations were first derived by von Karman and are usually referred to as *von Karman's large deflection equations*. In view of their complex nature, it is not possible to obtain a closed-form solution. Therefore, an approximate or a numerical method has to be applied. In the approximate method, a simple solution for $w(x, y)$ satisfying the boundary conditions is assumed. This function, when substituted in Eq. (8.11), leads to a differential equation in ϕ . The solution for ϕ is then substituted in Eq. (8.12) to obtain a better estimate for $w(x, y)$. This estimate, when substituted in Eq. (8.11), gives a better solution for ϕ . The procedure is terminated when the desired convergence has been attained. This process, though simple to look at, is quite involved since the differential equations are partial and nonhomogeneous.

EXAMPLE 8.1 (Postbuckling behaviour of simply-supported plate) Consider a square plate simply-supported on all sides (Fig. 8.2).

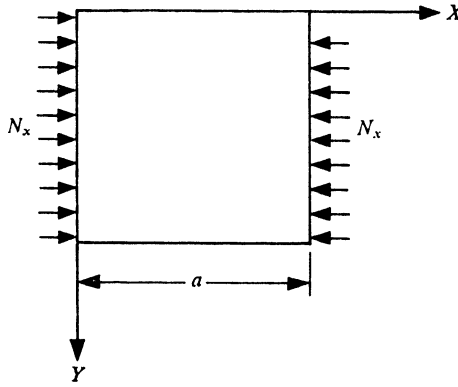


Fig. 8.2 Simply-Supported Square Plate under Uniaxial Compression.

In Chapter 5, we imposed the boundary conditions only on the transverse displacement $w(x, y)$. Here, since the equations are coupled, we need to prescribe, in addition, conditions on the inplane displacements. The conditions on $w(x, y)$ are

$$w = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0 \quad (x = 0, a), \quad (8.13a)$$

$$w = D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) = 0 \quad (y = 0, a). \quad (8.13b)$$

As the edges are supported on all sides, these equations reduce to

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad (x = 0, a), \quad (8.14a)$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \quad (y = 0, a). \quad (8.14b)$$

The plate is supported in such a manner that the inplane displacements u and v are not restricted. The possible inplane boundary conditions are

$$N_y = 0 \quad \text{or} \quad v = 0 \quad (y = 0, a), \quad (8.15a)$$

$$N_{xy} = 0 \quad (\text{along all edges}), \quad (8.15b)$$

$$N_x = 0 \quad \text{or} \quad u = 0 \quad (x = 0, a). \quad (8.15c)$$

Since the applied load is along the X -direction, u is not equal to zero, and since the displacement v is freely allowed, N_y must be zero. Further, N_{xy} is zero on all the edges. Thus, the inplane conditions reduce to

$$\begin{aligned} N_y &= 0 & (y = 0, a), \\ N_{xy} &= 0 & (x = 0, a, y = 0, a). \end{aligned} \quad (8.16)$$

In view of Eqs. (8.10), Eqs. (8.16) can be written as

$$\frac{\partial^2 \phi}{\partial x^2} = 0 \quad (y = 0, a), \quad (8.17a)$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad (x = 0, a, y = 0, a). \quad (8.17b)$$

The method of solution we follow here is based on the work of A.S. Volmir (see Chajes [1]) and K. Marguerre (see Bleich [2]).

In terms of the nondimensional parameters ξ and η , defined as

$$\xi = x/a, \quad \eta = y/b, \quad (8.18)$$

Eqs. (8.11) and (8.12) can be written as

$$\frac{\partial^4 \phi}{\partial \xi^4} + 2p^2 \frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 \phi}{\partial \eta^4} = Ep^2 \left[\left(\frac{\partial^4 w}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} \right], \quad (8.19)$$

$$\frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} = \frac{t}{D} p^2 \left(\frac{\partial^2 \phi}{\partial \eta^2} \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - 2 \frac{\partial^2 \phi}{\partial \xi \partial \eta} \frac{\partial^2 w}{\partial \xi \partial \eta} \right), \quad (8.20)$$

where $p (=a/b)$ is the aspect ratio of the plate. The boundary conditions, written also in terms of ξ and η , are

$$w(\xi, \eta) = \frac{\partial^2 w}{\partial \xi^2} \quad (\xi = 0, 1), \quad (8.21a)$$

$$w(\xi, \eta) = \frac{\partial^2 w}{\partial \eta^2} \quad (\eta = 0, 1), \quad (8.21b)$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = 0 \quad (\xi = 0, 1), \quad (8.21c)$$

$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0 \quad (\xi = 0, 1, \eta = 0, 1). \quad (8.21d)$$

Since we are considering a square plate with $p = 1$, we assume that, as the load reaches the initial buckling value, the plate buckles with one half sine wave along the ξ -direction and one half sine wave along the η -direction. This assumption may not be true for other values of p (see Section 5.2). Thus,

$$w(\xi, \eta) = A_1 \sin \pi \xi \sin \pi \eta. \quad (8.22)$$

This satisfies the boundary conditions (8.21a) and (8.21b) identically. Substituting relation (8.22) in Eq. (8.19), we obtain

$$\frac{\partial^4 \phi}{\partial \xi^4} + 2 \frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 \phi}{\partial \eta^4} = \frac{A_1^2 E \pi^4}{2} (\cos^2 \pi \xi \cos^2 \pi \eta - \sin^2 \pi \xi \sin^2 \pi \eta). \quad (8.23)$$

Using the trigonometric relations

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta, \quad \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta,$$

we find Eq. (8.23) reduces to

$$\frac{\partial^4 \phi}{\partial \xi^4} + 2 \frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 \phi}{\partial \eta^4} = \frac{A_1^2 E \pi^4}{2} (\cos 2\pi \xi + \cos 2\pi \eta). \quad (8.24)$$

This is a linear nonhomogeneous partial differential equation, and its solution must satisfy the boundary conditions (8.21c) and (8.21d). Since the applied load is uniform with respect to η , the complimentary solution of Eq. (8.24) can be written as

$$\phi_c = B_1 \eta^2. \quad (8.25)$$

The particular solution of Eq. (8.24) depends on the nonhomogeneous part of the equation. As this equation is linear, the particular solution can be written as

$$\phi_p = C_1 \cos 2\pi \xi + D_1 \sin 2\pi \eta. \quad (8.26)$$

Substituting expression (8.26) in Eq. (8.24) and equating the coefficients of the same type, we get

$$\phi_p = \frac{EA^2}{32} (\cos 2\pi \xi + \cos 2\pi \eta).$$

The complete solution is

$$\phi = \phi_p + \phi_c = \frac{EA_1^2}{32} (\cos 2\pi \xi + \cos 2\pi \eta) + B_1 \eta^2. \quad (8.27)$$

The constant B_1 is obtained from the conditions on stress distribution that exist prior to buckling. Since the stresses are $N_y = 0$, $N_{xy} = 0$, and $N_x = a$ constant, we have, prior to buckling,

$$N_x = -t \frac{\partial^2 \phi_p}{\partial y^2} = -\frac{t}{a^2} \frac{\partial^2 \phi_p}{\partial \eta^2}. \quad (8.28)$$

Substituting for ϕ_p from Eq. (8.26), we get

$$B_1 = -N_x \frac{a^2}{2t}. \quad (8.29)$$

Hence,

$$\phi = \frac{EA_1^2}{32}(\cos 2\pi\xi + \cos 2\pi\eta) - N_x \frac{a^2}{2t}\eta^2. \quad (8.30)$$

The constant A_1 is still an unknown. This can be obtained by substituting Eq. (8.30) in Eq. (8.12) and then solving for $w(\xi, \eta)$. This gives a highly complicated expression. As suggested by Chajes [1], an approximate solution to this expression can be obtained by using the Galerkin technique. In our context, such a solution is

$$\int_1^L \int_0^1 \bar{L}(\xi, \eta) g(\xi, \eta) d\xi d\eta = 0, \quad (8.31)$$

where $\bar{L}$, obtained from Eq. (8.20), is

$$\bar{L}(\xi, \eta) = [4A_1 D\pi^4 - \frac{A_1^3 E\pi^4 t}{8}(\cos 2\pi\xi + \cos 2\pi\eta) - N_x a^2 A_1 \pi^2] \sin \pi\xi \sin \pi\eta \quad (8.32)$$

and $g(\xi, \eta)$ is the mode shape for $w(\xi, \eta)$. In Eq. (8.31), substituting for $\bar{L}(\xi, \eta)$ from Eq. (8.32) and writing $g(\xi, \eta) = \sin \pi\xi \sin \pi\eta$ and integrating, we get

$$N_x = \frac{4D\pi^2}{a^2} + \frac{E_1 A_1^3 \pi t}{8a^2}. \quad (8.33)$$

In Eq. (8.33), the first term corresponds to the initial buckling load [see Eq. (5.54)], and the second term gives the additional load the plate can resist because of the finite deflection. Therefore, the average stress in the plate is given by

$$\sigma_{xa} = 4 \frac{D\pi^2}{a^2 t} + \frac{EA_1^3 \pi}{8a^2}. \quad (8.34)$$

Thus, the average stress in the plate in the postbuckling range is proportional to the square of the amplitude of deflection and increases as the deflection increases. As stated in Chapter 1, the behaviour of a plate is entirely different from that of a column. In the latter, the initial and postbuckling loads are the same, whereas those in the former are very much different.

The foregoing analysis is applicable for stress levels within the elastic limit. Even with this limitation, it is possible to obtain postbuckling loads which are very much higher than the initial buckling load. This fact is made use of when designing an aircraft structure where weight minimization is extremely important.

The longitudinal stress σ_x is

$$\sigma_x = -\frac{\partial^2 \phi}{a^2 \partial \eta^2} = -\frac{EA_1^2 \pi^2}{8a^2} \cos 2\pi\eta + \frac{N_x}{t}$$

or

$$\sigma_x = \sigma_{xa} + (\sigma_{xa} - \sigma_{cr}) \cos 2\pi\eta \quad (8.35)$$

and

$$\sigma_y = -\frac{1}{a^2} \frac{\partial^2 \phi}{\partial \xi^2} = \frac{EA_1^2 \pi^2}{8a^2} \cos 2\pi\xi$$

or

$$\sigma_y = (\sigma_{xa} - \sigma_{cr}) \cos 2\pi\xi. \quad (8.36)$$

Figure 8.3 shows the variation of σ_x and σ_y at the middle surface of the plate. We note

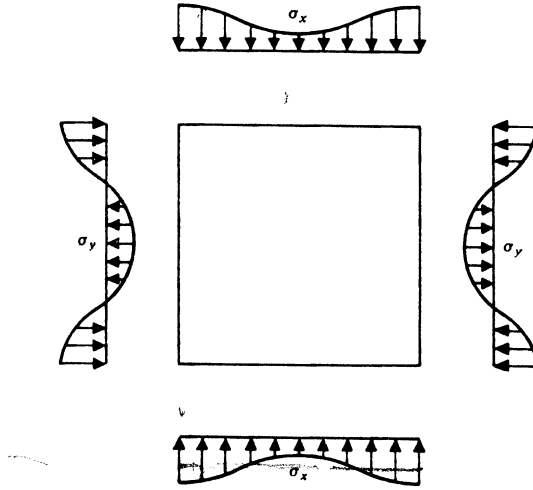


Fig. 8.3 Variation of Stresses in the Middle Surface of Plate.

from the figure that the stress σ_x which was constant across the width of the plate prior to buckling is no longer constant. Further, the ends of the plate withstand more stresses than the middle portion of the plate. This implies that, in the plate, the sides are stiffer than the middle portion. In fact, the change in stress in the middle portion before and after buckling is almost negligible. The stress σ_y is zero before buckling. However, after buckling, if the middle portion is constrained to remain straight, tensile stresses develop. It is these stresses that are responsible for the postbuckling strength of the plate. If the unloaded edges are not supported, then the postbuckling load is much smaller than that with supported edges. In practical situations, the edges are normally supported, for example, in sheet-stringer stiffened plate and web plate of I -section.

8.3 ENERGY APPROACH

As stated in Section 8.2, the differential equation approach becomes quite complicated when the postbuckling load of a plate has to be evaluated. In such a situation, as suggested

by Timoshenko and Gere [3], it is advisable that the Rayleigh-Ritz technique be applied. When using this approach, the deflection of the plate is determined from the condition of minimum strain energy. The strain energy of a plate is due to (i) the inplane stresses and (ii) the bending stresses.

Strain energy due to inplane stresses

The inplane stresses in a plate are σ_x , σ_y , and τ_{xy} . The strain energy U_m is then given by

$$U_m = \frac{t}{2} \int_{-a}^a \int_{-b}^b (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}) dx dy. \quad (8.37)$$

Using Eqs. (8.8), we can write U_m in terms of strains alone as

$$U_m = \frac{Et}{2(1-\nu^2)} \int_{-a}^a \int_{-b}^b (\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + \frac{1-\nu}{2}\gamma_{xy}^2) dx dy. \quad (8.38)$$

Substituting the strain-deformation relations (8.6), we can express Eq. (8.38) in terms of u_0 , v_0 , and w as

$$\begin{aligned} U_m = \frac{Et}{2(1-\nu^2)} \int_{-a}^a \int_{-b}^b [& (\frac{\partial u_0}{\partial x})^2 + \frac{1}{2}(\frac{\partial w}{\partial x})^4 + (\frac{\partial v_0}{\partial y})^2 + \frac{1}{2}(\frac{\partial w}{\partial y})^4 + \frac{\partial u_0}{\partial x}(\frac{\partial w}{\partial x})^2 + \frac{\partial v_0}{\partial y}(\frac{\partial w}{\partial y})^2 \\ & + 2\nu\{\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} + \frac{1}{2}\frac{\partial v_0}{\partial y}(\frac{\partial w}{\partial x})^2 + \frac{1}{2}(\frac{\partial u_0}{\partial x})(\frac{\partial w}{\partial y})^2 + \frac{1}{2}(\frac{\partial w}{\partial x})^2(\frac{\partial w}{\partial y})^2\} + (\frac{1-\nu}{2})\{(\frac{\partial v_0}{\partial x})^2 \\ & + (\frac{\partial u_0}{\partial y})^2 + (\frac{\partial w}{\partial x})^2(\frac{\partial w}{\partial y})^2 + 2(\frac{\partial v_0}{\partial x})(\frac{\partial u_0}{\partial y}) + 2\frac{\partial v_0}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + 2\frac{\partial u_0}{\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\}] dx dy. \end{aligned} \quad (8.39)$$

The strain energy due to bending is given by Eq. (5.99).

The total strain energy of a plate is given by

$$U = U_b + U_m. \quad (8.40)$$

The unknown constants in U_b and U_m are determined from the condition that the strain energy U is a minimum.

EXAMPLE 8.2 (Buckling load of rectangular plate) Consider a simply-supported plate compressed along the Y -direction and with no inplane displacement along the X -direction (see Fig. 8.4). Since the loading and the structure are symmetric, the coordinate system we choose is as shown in the figure.

An approximate expression of the type

$$w = A \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2b} \quad (8.41)$$

satisfies the boundary conditions exactly. The accuracy in this representation can be improved by taking additional terms in the series for w . (This is left to the reader as an exercise.) The boundary conditions for the inplane displacements u and v are

$$u = 0 \quad (x = \pm a, y = \pm b), \quad (8.42a)$$

$$v = \text{constant} \quad (x = \pm a, y = \pm b). \quad (8.42b)$$

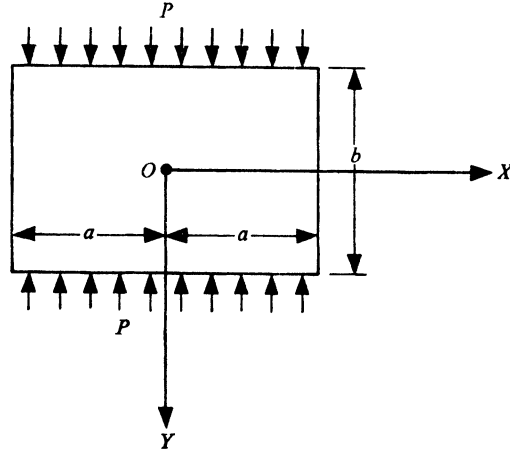


Fig. 8.4 Rectangular Plate under Axial Load.

The displacement functions that satisfy the inplane boundary conditions are

$$u_0 = c_1 \sin \frac{\pi x}{a} \cos \frac{\pi y}{2b}, \quad (8.43a)$$

$$v_0 = c_2 \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} - \epsilon y. \quad (8.43b)$$

The second term in v_0 satisfies the condition of uniform strain in the Y -direction. The expressions for strains ϵ_x , ϵ_y , and γ_{xy} in terms of u_0 , v_0 , and w are given by (8.6). Further, the strain energy due to inplane strains is given by Eq. (8.39). Substituting for u_0 , v_0 , and w in Eq. (8.39), we get

$$\begin{aligned} U_m = & \frac{Et}{2(1-\nu^2)} \int_{-a}^a \int_{-b}^b \left[c_1^2 \frac{\pi^2}{a^2} \cos^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{2b} + \frac{A^4}{4} \left(\frac{\pi}{2a} \right)^4 \sin^4 \frac{\pi x}{2a} \cos^4 \frac{\pi y}{2b} + \epsilon^2 \right. \\ & + c_2^2 \frac{\pi^2}{b^2} \cos^2 \frac{\pi x}{2a} \cos^2 \frac{\pi y}{b} - 2\epsilon c_2 \frac{\pi}{b} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{b} + \frac{A^4}{4} \left(\frac{\pi}{2b} \right)^4 \cos^4 \frac{\pi x}{2a} \sin^4 \frac{\pi y}{2b} \\ & + \frac{A^2 c_1 \pi^3}{4a^3} \cos \frac{\pi x}{a} \cos \frac{\pi y}{2b} \sin^2 \frac{\pi x}{2a} \cos^2 \frac{\pi y}{2b} + \frac{A^2 c_2 \pi^3}{4b^3} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{b} \cos^2 \frac{\pi x}{2a} \sin^2 \frac{\pi y}{2b} \\ & - \epsilon \frac{A^2 \pi^2}{4b^2} \cos^2 \frac{\pi x}{2a} \sin^2 \frac{\pi y}{2b} + 2\nu \{ c_1 c_2 \frac{\pi^2}{ab} \cos \frac{\pi x}{a} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2b} \cos \frac{\pi y}{b} \\ & - \epsilon c_1 \frac{\pi}{a} \cos \frac{\pi x}{a} \cos \frac{\pi y}{2b} + \frac{1}{2} A^2 c_2 \frac{\pi^3}{4a^2 b} \sin^2 \frac{\pi x}{2a} \cos^2 \frac{\pi y}{2b} \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \\ & \left. - \frac{1}{2} \epsilon A^2 \frac{\pi^2}{4a^2} \sin^2 \frac{\pi x}{2a} \cos^2 \frac{\pi y}{2b} + \frac{1}{2} A^2 \frac{\pi^3}{4ab^2} \cos^2 \frac{\pi x}{2a} \sin^2 \frac{\pi y}{2b} \cos \frac{\pi x}{a} \cos \frac{\pi y}{2b} \right] dx dy \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} A^4 \frac{\pi^4}{16a^2b^2} \sin^2 \frac{\pi x}{2a} \cos^2 \frac{\pi y}{2b} \cos^2 \frac{\pi x}{2a} \sin^2 \frac{\pi y}{2b} \} + \frac{(1-\nu)}{2} \{ c_2^2 \frac{\pi^2}{4a^2} \sin^2 \frac{\pi x}{2a} \sin^2 \frac{\pi y}{b} \\
& + c_1^2 \frac{\pi^2}{4b^2} \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{2b} + A^4 \frac{\pi^4}{16a^2b^2} \sin^2 \frac{\pi x}{2a} \cos^2 \frac{\pi y}{2b} \cos^2 \frac{\pi x}{2a} \sin^2 \frac{\pi y}{2b} \\
& + 2c_1c_2 \frac{\pi^2}{4ab} \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} \sin \frac{\pi x}{a} \sin \frac{\pi y}{2b} \\
& - 2A^2c_2 \frac{\pi^3}{8a^2b} \sin^2 \frac{\pi x}{2a} \cos \frac{\pi y}{2b} \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \sin \frac{\pi y}{2b} \\
& - 2A^2c_1 \frac{\pi^3}{8a^2b} \sin \frac{\pi x}{a} \sin \frac{\pi x}{2a} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2b} \sin^2 \frac{\pi y}{2b} \}] dx dy. \quad (8.44)
\end{aligned}$$

The bending energy U_b is obtained after substituting for w in Eq. (5.99). On integrating the result so derived, we get

$$U_b = \frac{\pi^4 ab A^2 D}{32} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2. \quad (8.45)$$

Now, integrating Eq. (8.44) and adding the resulting expression to Eq. (8.45), we get

$$\begin{aligned}
U = & \frac{\pi^4 ab A^2 D}{32} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2 + \frac{Et}{2(1-\nu^2)} [4ab\epsilon^2 - \frac{\pi^2 A^2 a\epsilon}{4b} + \frac{\pi^4 A^4}{1024ab} (9\frac{b^2}{a^2} + 9\frac{a^2}{b^2} + 2) \\
& - \frac{\pi^2 A^2}{6} \{ c_2 (\frac{2a}{b^2} + \frac{1-3\nu}{2a}) + c_1 (\frac{2b}{a^2} + \frac{1-3\nu}{2b}) \} + \pi^2 \{ c_2^2 (\frac{a}{b} + \frac{1-\nu}{8} \frac{b}{a}) \\
& + c_1^2 (\frac{b}{a} + \frac{1-\nu}{8} \frac{a}{b}) \} + c_1c_2 \frac{1}{9} (1+\nu) - \frac{\nu\pi^2 A^2 b\epsilon}{4a}]. \quad (8.46)
\end{aligned}$$

For a given value of ϵ , the constants c_1 , c_2 , and A are determined from the condition that the strain energy U is a minimum. Thus,

$$\frac{\partial U}{\partial A} = 0, \quad \frac{\partial U}{\partial c_1} = 0, \quad \frac{\partial U}{\partial c_2} = 0. \quad (8.47)$$

This results in

$$\begin{aligned}
& \frac{\pi^4 ab AD}{16} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2 + \frac{Et}{2(1-\nu^2)} \left[-\frac{\pi A a \epsilon}{2b} + \frac{\pi^4 A^3}{256ab} (9\frac{b^2}{a^2} + 9\frac{a^2}{b^2} + 2) \right. \\
& \left. - \frac{\pi^2 A}{3} \{ c_2 (\frac{2a}{b^2} + \frac{1-3\nu}{2a}) + c_1 (\frac{2b}{a^2} + \frac{1-3\nu}{2b}) \} - \frac{\nu\pi^2 A b \epsilon}{2a} \right] = 0, \quad (8.48a)
\end{aligned}$$

$$-\frac{\pi^2 A^2}{6} \left(\frac{2b}{a^2} + \frac{1-3\nu}{2b} \right) + 2\pi^2 c_1 \left(\frac{b}{a} + \frac{1-\nu}{8} \frac{a}{b} \right) + c_2 \frac{1}{9} (1+\nu) = 0, \quad (8.48b)$$

$$-\frac{\pi^2 A^2}{6} \left(\frac{2a}{b^2} + \frac{1-3\nu}{2a} \right) + 2\pi^2 c_2 \left(\frac{a}{b} + \frac{1-\nu}{8} \frac{b}{a} \right) + c_1 \frac{1}{9} (1+\nu) = 0. \quad (8.48c)$$

Since, for a square plate, $a = b$, $c_1 = c_2 = c$, we get, from Eqs. (8.48b) and (8.48c),

$$\frac{A^2}{a}(-4.112 + 2.467\nu) + c(23.984 - 0.690\nu) = 0. \quad (8.49)$$

Since, for most engineering materials, $\nu = 0.3$, Eq. (8.49) reduces to

$$c = (0.1418)\frac{A^2}{a}. \quad (8.50)$$

Substituting this value of c in Eq. (8.48a) for c_1 and c_2 , we get

$$\begin{aligned} \frac{\pi^4 a^2 AD}{16} \frac{4}{a^4} + \frac{Et}{2(1-\nu^2)} \left[-\frac{\pi A \epsilon}{2} + \frac{\pi^4 A^3}{256 a^2} \times 20 \right. \\ \left. - \frac{\pi^2 A}{3} \times 0.1418 \frac{A^2}{a} \left(\frac{4}{a} + \frac{1-3\nu}{a} \right) - \nu \frac{\pi^2 A}{2} \epsilon \right] = 0 \end{aligned}$$

or

$$A(4.06t^2 - 6.42a^2\epsilon + 5.688A^2) = 0. \quad (8.51)$$

For a nontrivial solution, $A \neq 0$; hence, for a solution to exist, the quantity within the parentheses must be zero. Thus,

$$A = \sqrt{6.42a^2\epsilon - 4.06t^2}. \quad (8.52)$$

Here, since A is real, only the positive sign is retained. Further, for A to be real, the quantity under the radical sign must be zero. This gives ϵ_{cr} as

$$\epsilon_{cr} = (0.632)\frac{t^2}{a^2}. \quad (8.53)$$

This is the strain at which there is a finite deflection.

The compressive load on the plate is

$$P = -t \int_{-b}^b \sigma_y dx, \quad (8.54)$$

$$\sigma_y|_{y=b} = \frac{E}{(1-\nu^2)}[(\epsilon_y)_{y=b} + \nu(\epsilon_x)_{y=b}]. \quad (8.55)$$

The inplane strains ϵ_x and ϵ_y at $y = b$ from Eqs. (8.6), after substitution for u_0 , v_0 , and w , are

$$\epsilon_x|_{y=b} = 0, \quad (8.56a)$$

$$\epsilon_y|_{y=b} = -\frac{\pi A^2}{a^2}(0.418) \cos \frac{\pi x}{2a} - \epsilon + \frac{A^2}{2} \frac{\pi^2}{4a^2} \cos^2 \frac{\pi x}{2a}. \quad (8.56b)$$

Substituting for ϵ_{cr} from relation (8.53), we get

$$\sigma_y|_{y=b} = \frac{E}{1-\nu^2} \left[-\frac{\pi A^2}{a^2} \times 0.418 \cos \frac{\pi x}{2a} - 0.632 \frac{t^2}{a^2} + \frac{A^2}{2} \frac{\pi^2}{4a^2} \cos^2 \frac{\pi x}{2a} \right]. \quad (8.57)$$

Hence, for a square plate (i.e., $a = b$),

$$P = -t \int_{-a}^a \frac{E}{(1-\nu^2)} \left[-(0.418) \frac{\pi A^2}{a^2} \cos \frac{\pi x}{2a} - (0.632) \frac{t^2}{a^2} + \frac{A^2}{2} \frac{\pi^2}{4a^2} \cos^2 \frac{\pi x}{2a} \right] dx$$

$$= -\frac{Et}{1-\nu^2} \left[-(1.672) \frac{A^2}{a} - (0.632) \frac{t^2}{a} + \frac{A^2 \pi^2}{8a} \right]. \quad (8.58)$$

For more details on the postbuckling behaviour of plates, see Bleich [2].

8.4 ULTIMATE COMPRESSIVE STRENGTH OF FLAT SHEETS

The foregoing analyses show that a flat sheet (plate) does not fail when the initial buckling load is reached. This is because the stiffening members are usually capable of resisting stresses higher than those corresponding to the initial buckling load. The ultimate load is, to a large extent, governed by the failure of a stiffening member itself. For example, in a rectangular plate simply-supported on all sides, the stress distribution across the width of the plate varies as schematically shown in Fig. 8.5. Here, the stress distribution across the width of the plate is indicated by lines 1 and 2. Line 2 gives the stress level at which the initial buckling takes place. As the applied load increases beyond this value, the stress distribution is as given by lines 3, 4, 5, and 6. It can be observed that the stress in the central region of the plate hardly increases beyond the initial buckling stress.

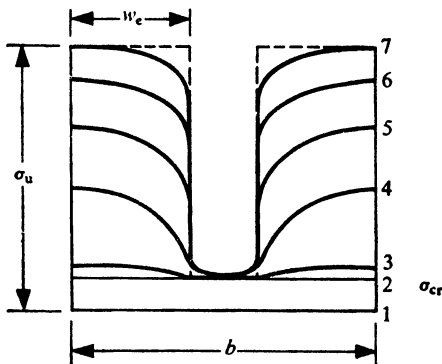


Fig. 8.5 Schematic Variation of Stress across Width of Plate.

Therefore, the plate can be thought of as composed of two strips, each of width, say, w_e , on which a uniform stress σ_u acts. The total load carried by the plate is obtained from σ_u .

To obtain the approximate value of w_e , assume that the effective width of the plate is $2w_e$, the plate being simply-supported on all sides and with buckling stress σ_u . From Eq. (5.59), σ_u is

$$\sigma_u = KE \left(\frac{t}{2w_e} \right)^2, \quad (8.59)$$

where $K = 3.62$. The effective width w_e then is

$$w_e = 0.95t \sqrt{\frac{E}{\sigma_u}}. \quad (8.60)$$

The numerical factor in Eq. (8.60) is slightly higher than that determined experimentally. A relation of the type

$$w_e = 0.85t \sqrt{\frac{E}{\sigma_u}} \quad (8.61)$$

satisfies the experimental data fairly accurately. Further, the effective width obtained by using Eq. (8.61) is lower than that calculated by exact analysis. However, from the design point of view, this is quite safe since it represents a conservative estimate. For a sheet supported on three sides and free on the fourth side,

$$w_e = 0.60t \sqrt{\frac{E}{\sigma_u}} \quad (8.62)$$

On the basis of the analysis carried out for different values of the aspect ratio of a plate, Argyris and Dunne [4] have suggested several ways of obtaining the effective width. For a discussion on these methods, the interested reader may see Chajes [1], Bleich [2], Timoshenko and Gere [3], and Rivello [5]. Further, the method suggested by Peery [6] is also fairly accurate.

EXAMPLE 8.3 (Buckling of sheet-stringer panel) Consider an axially loaded aluminium sheet-stringer panel simply-supported along the loaded edges and at the rivet lines and free at the sides (Fig. 8.6). Each stringer has an area 0.6×10^{-4} sq m. We further suppose that

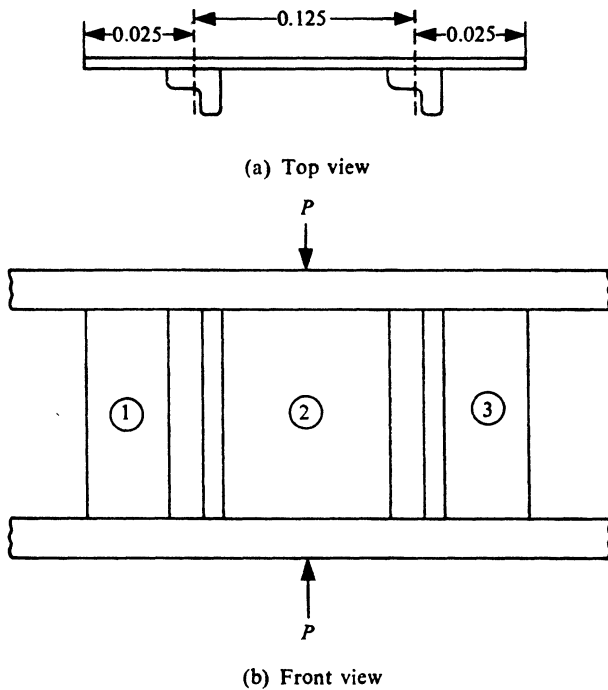


Fig. 8.6 Sheet-Stringer Panel.

panel 2 has the dimensions $0.25 \text{ m} \times 0.125 \text{ m} \times 0.001 \text{ m}$ and is simply-supported on all sides, and that each of panels 1 and 3 has the dimensions $0.25 \text{ m} \times 0.025 \text{ m} \times 0.001 \text{ m}$ and is

simply-supported on three sides and free on the fourth side. Taking $E = 0.588 \times 10^{11}$ N/m², we shall determine the compressive load

- (i) when the sheet first buckles,
 - (ii) when the stringer stress $\sigma_u = 58.84 \times 10^6$ N/m².
- (i) The initial critical stress σ_c for panels 1, 3 and 2 is

$$\begin{aligned}\sigma_{c1,3} &= KE(t/b)^2 \\ &= 0.385 \times 0.588 \times 10^{11} \times \left(\frac{0.001}{0.025}\right)^2 \\ &= 3.53 \times 10^7 \text{ N/m}^2, \\ \sigma_{c2} &= 3.62 \times 0.588 \times 10^{11} \times \left(\frac{0.001}{0.125}\right)^2 \\ &= 1.363 \times 10^7 \text{ N/m}^2.\end{aligned}$$

Thus, the sheet between the stringers buckles first. Just prior to this buckling, the entire width of the plate is effective in carrying the load. The area of the plate

$$A = 0.175 \times 0.001 + 2 \times 0.6 \times 10^{-4} = 2.95 \times 10^{-4} \text{ sq m},$$

and therefore

$$P = \sigma_c A = 2.95 \times 10^{-4} \times 1.363 \times 10^7 = 4020.85 \text{ N}.$$

(ii) For the postbuckling region, the load is calculated by using the effective width obtained by applying Eqs. (8.61) and (8.62). Thus,

$$\begin{aligned}w_{e1} &= 0.85t \sqrt{\frac{E}{\sigma_u}} \\ &= 0.85 \times 0.001 \sqrt{[0.588 \times 10^{11} / (58.84 \times 10^6)]} \\ &= 0.02688 \text{ m}, \\ w_{e2} &= 0.6t \sqrt{\frac{E}{\sigma_u}} \\ &= 0.6 \times 0.001 \sqrt{[0.588 \times 10^{11} / (58.84 \times 10^6)]} \\ &= 0.01897 \text{ m}.\end{aligned}$$

The effective area of the sheet is

$$\begin{aligned}A &= 2(w_{e1} + 2w_{e2})t \\ &= 0.917 \times 10^{-4} \text{ sq m}.\end{aligned}$$

The total compressive load is

$$\begin{aligned} P &= \sigma_u(A_c + 2 \times 0.6 \times 10^{-4}) \\ &= 58.84 \times 10^6(0.917 + 1.2) \times 10^{-4} \\ &= 12456 \text{ N.} \end{aligned}$$

As can be observed, the postbuckling load is three times the initial buckling load.

8.5 ULTIMATE STRENGTH OF FLAT SHEETS IN SHEAR

The postbuckling behaviour of a sheet subjected to shear loading (extensively investigated by Kuhn et al [7] and Van der Neut [8]) is similar to that of an axially compressed sheet. Examples of such a sheet are thin web of a spar and web of an *I*-section girder. In wing spars, which are not exposed to air stream, the webs at times are permitted to buckle at a small fraction of the ultimate load. This results in a minimum weight structure.

In 1928, Hans Wagner demonstrated that a thin web stiffened by vertical members subjected to shear does not fail when the initial buckling load is reached. This is so because the applied load is transferred as a series of tension diagonals while the stiffeners take the compressive load. In arriving at his conclusion, Wagner assumed that the web is so thin that it cannot resist any compressive load. However, in a real-life structure, the webs do have some thickness, and hence a part of the applied load is resisted as shear stress and the remaining part as tensile stress. In order to appreciate and understand the behaviour of a tension field beam, it is worthwhile to know how a complete tension field beam behaves (see Peery [6] and Kuhn [9]).

8.5.1 Complete Tension Field Beam

Consider a beam with a very thin web (see Fig. 8.7). Assume that the entire bending load is

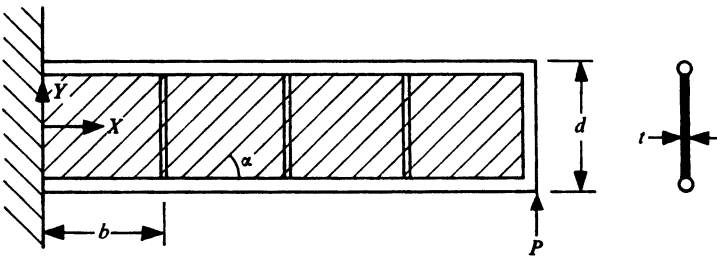


Fig. 8.7 Diagonal Tension Beam.

resisted by the flanges and the web resists only the shear. Also suppose that the web buckles into a series of diagonal elements, each of width unity. If we assume that the beam behaves like a complete tension field beam, then the stresses acting on a diagonal element are as shown in Fig. 8.8.

For the given beam, the shear force is constant along the length. Further, the shear flow q acting on the sheet is P/h and the shear stress at all points in the sheet is $P/(ht)$.

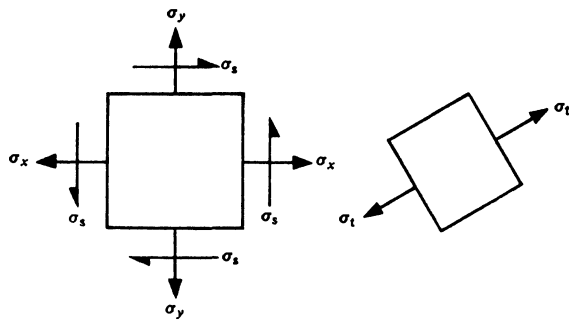


Fig. 8.8 Stresses Acting on Element.

The Mohr circle and the principal stress diagram for the shear stress distribution are shown in Fig. 8.9. This figure shows that this stress distribution results in σ_c and σ_t . If the

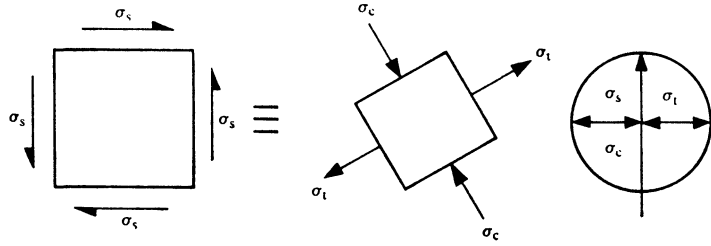


Fig. 8.9 Mohr's Circle and Principal Stresses.

web cannot resist any compressive stress, then the principal stress diagram reduces to the one in Fig. 8.10. The consequence of this assumption is that we need to introduce additional

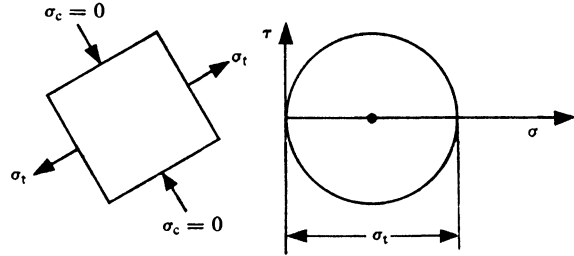


Fig. 8.10 Mohr's Circle and Principal Stress Diagram.

normal stresses σ_x and σ_y along the X - and Y -direction. From the geometry of Mohr's circle (Fig. 8.10),

$$\sigma_x = \sigma_s \cot \alpha, \tag{8.63}$$

$$\sigma_y = \sigma_s \tan \alpha, \quad (8.64)$$

$$\sigma_t = \frac{2\sigma_s}{\sin 2\alpha}. \quad (8.65)$$

These additional stresses σ_x , σ_y generated get transferred to the flanges and vertical stiffeners. Thus, the flanges, in addition to the axial stresses they carry, have to resist the distributed stress σ_y . Further, if the beam is shear resistant, no load is carried by the vertical stiffeners. Otherwise, the vertical stiffeners have to resist an additional compressive load equal to $\sigma_y tb$. The stress σ_x acts as a uniformly distributed load on the vertical members. This stress is not of much consequence to the central stiffeners because it acts on either side of these stiffeners. But for the vertical member at the end of the beam, this stress acts as an additional load. Thus, we observe that the flanges and vertical stiffeners have to be of a large cross-section to withstand the stresses that arise as a consequence of complete tension field assumption.

In our discussion, we have assumed that the vertical stiffeners are symmetrically placed on either side of the sheet (i.e., web). If this is not so, then the compressive load, because of the stress σ_y , acts eccentrically on the vertical stiffeners. As a result, not only the compressive stresses, but also the bending stresses, develop in the stiffeners.

It is well-known that if a sheet is completely shear resistant, then the angle α at which the wrinkles are formed is 45° . However, when the stresses σ_x and σ_y act, the value of α depends on the relative stiffness of the stiffener and the flange. For a technique to obtain α , see Peery [6].

8.5.2 Angle of Diagonal Tension

The beam shown in Fig. 8.7 is initially horizontal and when it is subjected to a tip load P , it undergoes a displacement along the Y -direction. This corresponds to a shear angle γ which is constant along the length of the beam. The angle α , as already noted, is due to the shear buckling of the panel of the web (see Fig. 8.7). Hence, we shall consider only the shear deformation and neglect the bending deflection. The shear deformation γ is due to (see Fig. 8.11) the elongation of the (i) vertical webs and flanges and (ii) web diagonals. To find the contribution of each of these, consider a panel of length d' ($=d \cot \alpha$) and depth d . The horizontal length of the panel is slightly different from d' . This difference is negligible provided γ turns out to be small. Assume that ϵ_x , ϵ_y , and ϵ are the unit axial strains of the vertical, horizontal, and shear members, respectively. The deformation due to a vertical stiffener is $\epsilon_y d$ (Fig. 8.11a). Hence, the shear angle γ_1 is

$$\gamma_1 = \frac{\epsilon_y d}{d \cot \alpha} = \epsilon_y \tan \alpha. \quad (8.66)$$

Further, the axial deformation of the flange is $\epsilon_x d \cot \alpha$ (Fig. 8.11b). Hence, the shear angle γ_2 is

$$\gamma_2 = \epsilon_x \frac{d \cot \alpha}{d} = \epsilon_x \cot \alpha. \quad (8.67)$$

To understand the deformation contribution due to the web, consider Fig. 8.11c. Here, the length AB of the diagonal is $d/\sin \alpha$. Hence, the axial deformation CD is $ed/\sin \alpha$. Therefore,

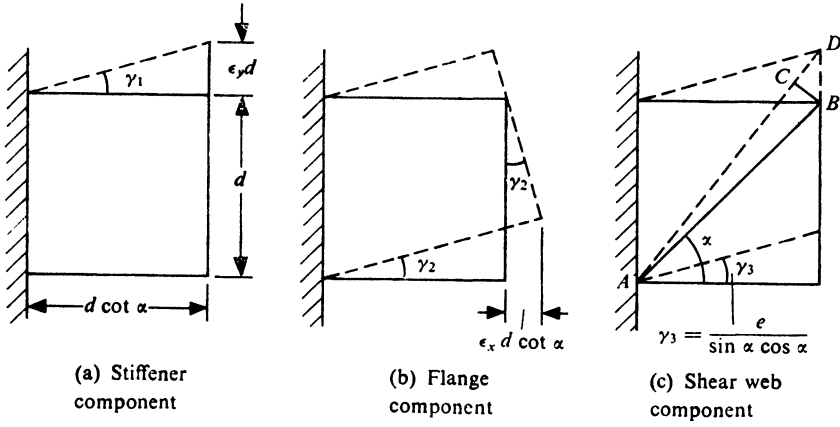


Fig. 8.11 Components of Shear Deformation.

the deformation BD is

$$BD = ed/\sin^2 \alpha.$$

Thus, the shear strain γ_3 is

$$\gamma_3 = \frac{ed}{\sin^2 \alpha} / (d \cot \alpha) = \frac{e}{\sin \alpha \cos \alpha}. \quad (8.68)$$

The total deformation γ is the algebraic sum of the components γ_1 , γ_2 , and γ_3 . Thus,

$$\gamma = -\epsilon_y \tan \alpha - \epsilon_x \cot \alpha + \frac{e}{\sin \alpha \cos \alpha}. \quad (8.69)$$

The negative sign here indicates that the axial strains in the vertical stiffeners and flange members are generally compressive.

The angle α is evaluated from the condition that the shear deformation γ is minimum. This implies

$$\frac{d\gamma}{d\alpha} = 0. \quad (8.70)$$

From Eq. (8.69), $d\gamma/d\alpha$ is

$$\frac{d\gamma}{d\alpha} = -\frac{\epsilon_y}{\cos^2 \alpha} + \frac{\epsilon_x}{\sin^2 \alpha} + e \left(\frac{1}{\cos^2 \alpha} - \frac{1}{\sin^2 \alpha} \right) = 0$$

or

$$\sin^2 \alpha (e - \epsilon_y) = \cos^2 \alpha (e - \epsilon_x)$$

or

$$\tan^2 \alpha = \frac{e - \epsilon_x}{e - \epsilon_y}. \quad (8.71)$$

The strains that develop in the members depend on the stresses which, in turn, depend on the load applied and the cross-sectional areas of the members. The stress in a sheet along the principal direction is σ_t . Hence, the axial strain e in the web is σ_t/E or $2\sigma_s \operatorname{cosec} 2\alpha/E$. Similarly, the strain ϵ_y is $-\sigma_s b \tan \alpha/(A_e E)$, where A_e is the effective area of the stiffener. If we assume that the axial strain due to the flange members is negligible (this is generally so since the flange members are quite stiff), then

$$\cot^4 \alpha = \frac{tb}{A_e} + 1. \quad (8.72)$$

The effective area of the stiffener depends on the structural configuration. If the stiffeners are placed on either side of a web, then A_e is equal to the actual cross-sectional area of the stiffeners. On the other hand, if the stiffeners are attached to only one side of a web, the load P due to σ_y will not only cause compressive stresses but also bending stresses which, in turn, depend on the distance between the centre of gravity of the load and that of the section. Thus, the compressive stress is

$$\sigma = \frac{\sigma_y t d}{A} + \frac{(\sigma_y t d) e^2}{I} = \frac{\sigma_y t d}{A} \left[1 + \frac{e^2}{k^2} \right] = \frac{\sigma_y t d}{A_e}, \quad (8.73)$$

where A_e is

$$A_e = \frac{A}{1 + e^2/k^2}. \quad (8.74)$$

8.5.3 Semi-Tension Field Beam

In Section 8.5.2, we assumed that the skin (i.e., web) is not capable of resisting any diagonal compressive load. This is generally not true since the web has some thickness. Therefore, the web resists the load partly as a shear resistant web and partly as a tension field web. A beam consisting of such a web is referred to as a semi-tension field beam. Examples of this beam are incomplete tension field beam and spar web.

Our analysis of the semi-tension field beam here is based on an engineering approach. Assume that the web shares a part of the applied load as a complete tension field beam and the remaining part of the load as a shear resistant beam. If P is the applied load, then $k_t P$ is the load resisted as the complete tension field beam and $(1 - k_t)P$ as the shear resistant web. The diagonal tension factor k_t is determined from an empirical relation. One such relation, derived experimentally by Kuhn [9], is

$$k_t = \tanh \left(0.5 \log_{10} \frac{\sigma_s}{\sigma_{scr}} \right), \quad (8.75)$$

where σ_s is the shear stress due to the applied load and σ_{scr} is the critical shear stress for a panel whose size is the same as that of a panel of a semi-tension field web. Once k_t is known from Eq. (8.75), we can find the distribution of stresses, and hence σ_x and σ_y .

Figure 8.12 (top and middle) schematically shows the stresses acting on an element of

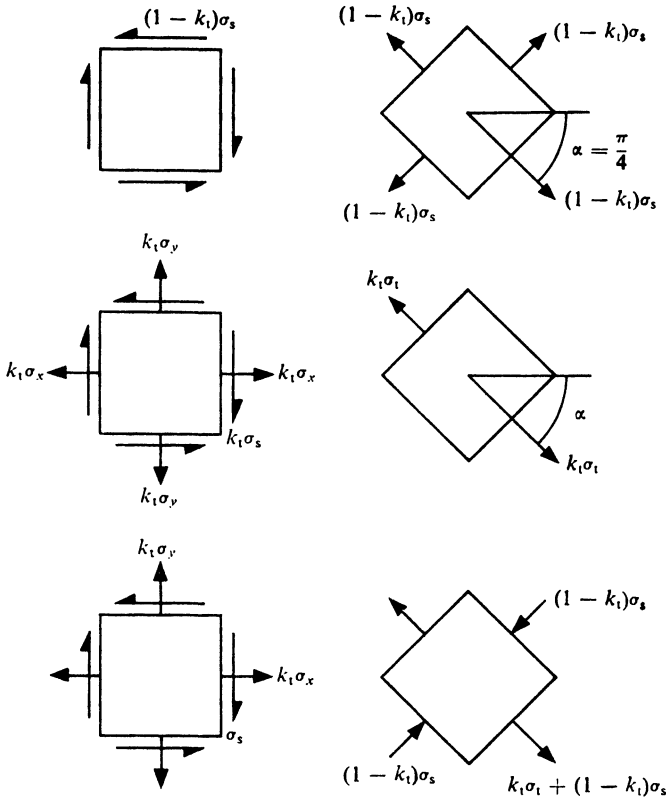


Fig. 8.12 Stresses Acting on Element.

the web for the two parts of the applied load; the bottom of the figure shows the stresses resulting from the total applied load.

For more details on semi-tension field beams, see Kuhn and Peterson [10].

EXAMPLE 8.4 (Axial loading in tension field beam) Consider the complete tension field web shown in Fig. 8.13. Assume that the buckling angle $\alpha = 45^\circ$. Now,

$$\sigma_s = \frac{P}{ht} = \frac{4000}{0.4 \times 0.001} = 10^7 \text{ N/m}^2.$$

Further, from Eqs. (8.63), (8.64), and (8.65),

$$\sigma_x = \sigma_s \cot \alpha = 10^7 \cot 45^\circ = 10^7 \text{ N/m}^2,$$

$$\sigma_y = \sigma_s \tan \alpha = 10^7 \text{ N/m}^2,$$

$$\sigma_t = \frac{2\sigma_s}{\sin 2\alpha} = \frac{2 \times 10^7}{\sin 90^\circ} = 2 \times 10^7 \text{ N/m}^2.$$

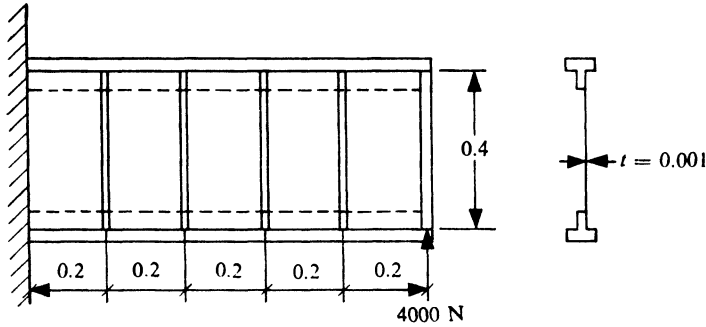


Fig. 8.13 Tension Field Beam.

The shear flow in the web is

$$q_s = \sigma_s t = 10^7 \times 0.001 = 10^4 \text{ N/m.}$$

The central stiffeners are subjected to the stress σ_x on both sides. However, the end stiffeners are subjected to the stress σ_x on only one side. Further, the stress σ_y acts on the flanges as a uniformly distributed load. The resultant of σ_x and σ_y is the compressive load on the vertical stiffeners.

On the flange members, there is a shear flow q_s ($= 10^4 \text{ N/m}$) in addition to σ_y , and at the ends of the flange members, there is an axial load due to the bending moment arising from the applied load P .

Figure 8.14 schematically shows the loads acting on the flange members, central stiffeners, and end stiffeners.

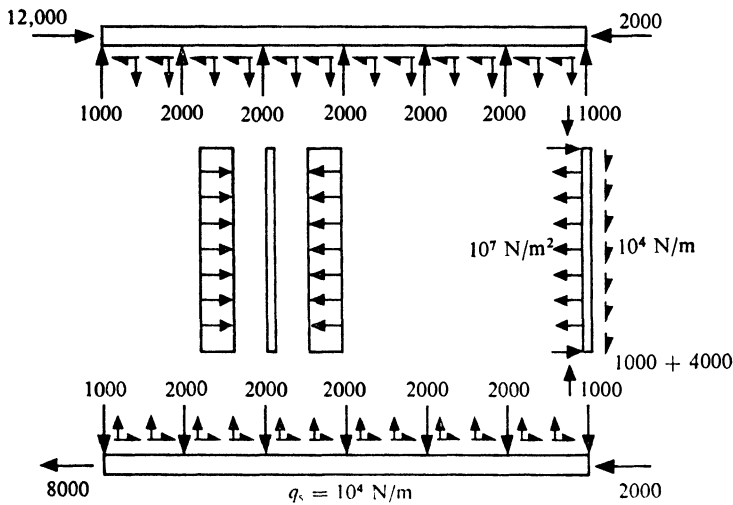


Fig. 8.14 Schematic Distribution of Load.

EXAMPLE 8.5 (Loads in semi-tension field beam) Figure 8.15 shows a semi-tension field beam. The stiffener is attached to only one side of the web, its cross-sectional area and second moment of area being $0.62 \times 10^{-4} \text{ m}^2$ and $0.7 \times 10^{-8} \text{ m}^4$, respectively. The thickness of the web is 0.001 m.

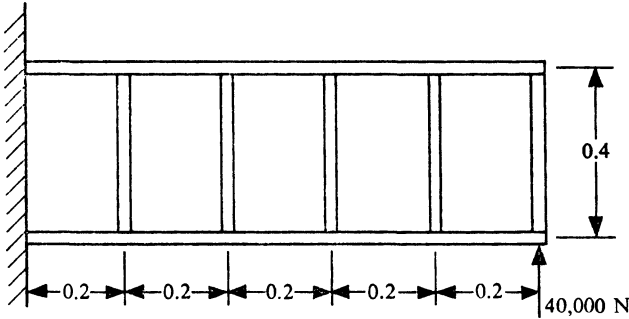


Fig. 8.15 Semi-Tension Field Beam under Tip Load.

As discussed in Section 5.4, the buckling stress for the web can be obtained from Eq. (5.152). Further, since the flanges and the vertical stiffeners are rivetted to the web, the fixity coefficient for the sheet will lie between the fixity coefficients for the simply-supported and clamped edges of the sheet. For simplicity, we shall work with the average of the fixity coefficient values for the two extreme cases.

Following Timoshenko and Gere [3], we have, for all edges simply-supported,

$$K = 4.34 + \frac{4.0}{(a/b)^2} \quad (8.76)$$

and, for all edges clamped,

$$K = 8.98 + \frac{5.6}{(a/b)^2}. \quad (8.77)$$

Therefore, for this web with $a/b = 2.0$, we get $K = 5.34$ for simply-supported edges and $K = 10.38$ for clamped edges. Thus, we shall use the average of these two values, i.e., $K = 7.86$, in our calculations. Assuming that the web is of an aluminium alloy with elastic modulus $E = 58.8 \times 10^9 \text{ N/m}^2$, we get

$$\sigma_{\text{scr}} = 7.86 \times 58.8 \times 10^9 \times \left(\frac{0.001}{0.2}\right)^2 = 1.156 \times 10^7 \text{ N/m}^2.$$

The shear stress σ_s is obtained as

$$\sigma_s = \frac{P}{ht} = \frac{40,000}{0.4 \times 0.001} = 10^8 \text{ N/m}^2.$$

Since the stiffener is attached to only one side of the sheet, we need to estimate the effective

area A_e of the stiffener. The stiffener radius of gyration is

$$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.7 \times 10^{-8}}{0.62 \times 10^{-4}}} = 1.06 \times 10^{-2}.$$

Let the eccentricity $e = 0.001$ m. The effective area A_e then is

$$A_e = \frac{0.7 \times 10^{-8}}{1 + \left(\frac{0.001}{1.06 \times 10^{-2}}\right)^2} = 0.37 \times 10^{-4} \text{ m}^2.$$

Thus,

$$\frac{td}{A_e} = \frac{0.001 \times 0.2}{0.37 \times 10^{-4}} = 5.4.$$

Therefore, the diagonal tension factor k_t is

$$k_t = \tanh \left(0.5 \log_{10} \frac{10^8}{1.156 \times 10^7} \right) = \tanh (0.5 \log_{10} 8.65) \\ = 0.43.$$

From Eq. (8.72),

$$\cot^4 \alpha = \frac{tb}{A_e} + 1 = 6.4.$$

Hence,

$$\alpha = 32.16^\circ,$$

$$\tan \alpha = 0.6287.$$

The stresses σ_y , σ_x are obtained as

$$\sigma_y = k_t \sigma_s \tan \alpha = 0.43 \times 10^8 \times 0.6287 = 2.703 \times 10^7 \text{ N/m}^2,$$

$$\sigma_x = k_t \sigma_s \cot \alpha = 0.43 \times 10^8 \times 1.59 = 6.579 \times 10^7 \text{ N/m}^2.$$

PROBLEMS

8.1 The skin on the upper side of an aircraft wing is made of 24-ST Alclad material. Further, the stringer spacing is 0.12 m and the rib spacing is 0.5 m. Assuming the edges to be simply-supported between the ribs and stringers, find the compression buckling stress for the skin gauges (i) 5×10^{-4} m, (ii) 10^{-3} m, and (iii) 1.5×10^{-3} m.

8.2 If, in Problem 8.1, the stringers are attached as in Fig. 8.16, what would be the compressive load

- (i) when the sheet first buckles?
- (ii) when the stringer stress is 62.78 MPa?
- (iii) when the stringer stress is 188.34 MPa?

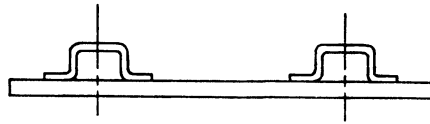


Fig. 8.16 Problem 8.2.

Assume that the sheet undergoes large deformation.

8.3 Figure 8.17 shows a sheet-stringer panel. It is subjected to a uniform axial compression.

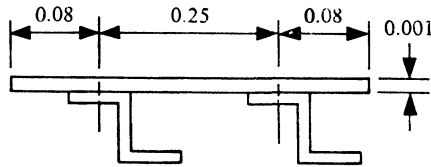


Fig. 8.17 Problem 8.3.

The sheet is assumed to be simply-supported at the loaded ends and along the rivet lines. The sides of the end panels are free. Each stringer has an area 1.2×10^{-4} sq m. Find the total compressive load P when the (i) sheet first buckles and (ii) stringer stress is 196.2 MPa. Assume $E = 70.63$ GPa. All dimensions are in metres.

8.4 For the semi-tension field beam shown in Fig. 8.18, compute the flange and vertical

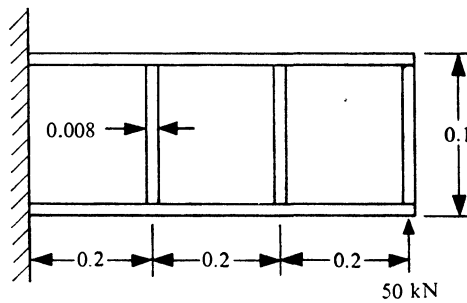
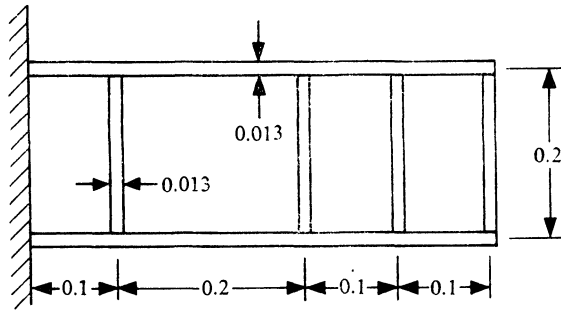


Fig. 8.18 Problem 8.4.

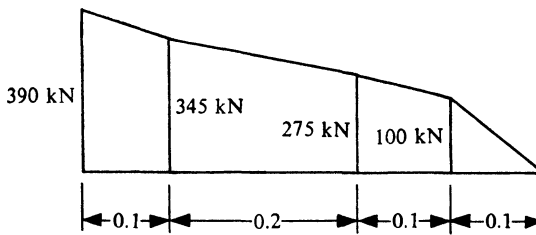
stiffener stresses. The stiffeners are placed on both sides of the sheet. The thickness of the sheet is 0.002 m and that of each of the stiffeners is 0.004 m.

8.5 Figure 8.19a shows a complete tension field beam. The bending moment varies along the length of the beam (see Fig. 8.19b). Further, a stiffener is attached on either side of the beam. Determine the stresses in the flange and the vertical stiffeners if the thickness of the

skin is 0.0013 m, the width of the flange is 0.003 m, and the thickness of each stiffener is 0.0015 m. All dimensions are in metres.



(a)



(b)

Fig. 8.19 Problem 8.5.

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Appendix I

Constitutive Equations for Anisotropic Materials

The generalized Hooke law relating stresses to strains can be written in the contracted notation as

$$\sigma_i = C_{ij}\epsilon_j \quad (i, j = 1, \dots, 6). \quad (\text{I.1})$$

In the tensor notation, it is written as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \quad (i, j, k, l = 1, 2, 3). \quad (\text{I.2})$$

Table I.1 shows the comparison between the tensor and contracted notations for stresses and strains. A general elastic material has therefore to be defined by 36 independent

Table I.1 Comparison between Tensor and Contracted Notations for Stresses and Strains

Stress		Strain	
Tensor notation	Contracted notation	Tensor notation	Contracted notation
σ_{11}	σ_1	ϵ_{11}	ϵ_1
σ_{22}	σ_2	ϵ_{22}	ϵ_2
σ_{33}	σ_3	ϵ_{33}	ϵ_3
$\tau_{23} = \sigma_{23}$	σ_4	$\gamma_{23} = 2\epsilon_{23}$	ϵ_4
$\tau_{31} = \sigma_{31}$	σ_5	$\gamma_{31} = 2\epsilon_{31}$	ϵ_5
$\tau_{12} = \sigma_{12}$	σ_6	$\gamma_{12} = 2\epsilon_{12}$	ϵ_6

constants. However, on the basis of strain energy, all the 36 constants are not independent. Consider an elastic material subjected to some external load. The work done per unit volume is

$$W_e = \frac{1}{2}C_{ij}\epsilon_i\epsilon_j. \quad (\text{I.3})$$

Thus,

$$\frac{\partial W_e}{\partial \epsilon_i} = C_{ij} \epsilon_j. \quad (1.4)$$

Hence,

$$\frac{\partial^2 W_e}{\partial \epsilon_i \partial \epsilon_j} = C_{ij}. \quad (1.5)$$

Similarly, it can be shown that

$$\frac{\partial^2 W_e}{\partial \epsilon_j \partial \epsilon_i} = C_{ji}. \quad (1.6)$$

Since the order of differential is immaterial,

$$C_{ij} = C_{ji}. \quad (1.7)$$

This implies that the off diagonal terms are symmetric. As a result, the number of constants reduces to 21. The stress-strain relationship can then be written in matrix notation as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}. \quad (1.8)$$

A material whose constitutive equations are given by (1.8) is referred to as an anisotropic material. Most materials, in general, are anisotropic.

If a material has one plane of symmetry (i.e., $Z = 0$), the constitutive equations (1.8) reduce to (see Love [1])

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}. \quad (1.9)$$

Such a material is called a *monoclinic material* and has 13 independent constants. Here, the normal stresses are related to the normal strains and to one of the shear strains. This implies that if normal stresses are applied to the material, the deformation then would be due to the normal and shear strains.

If a material has two orthogonal planes of symmetry, then the constitutive equations

(referred to the principal material directions) are

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}. \quad (\text{I.10})$$

Equations (I.10) define the stress-strain relations for a three-dimensional *orthotropic material* (e.g., wood and reinforced cement). As can be seen, in such a material, there is no interaction between the normal stresses and the shearing strains. Further, the number of independent constants is only nine.

If at every point in a material there is one plane along which the mechanical properties are the same in all directions, then the material is termed *transversely isotropic*. In such a material, if (1-2)-plane is the plane of symmetry, then the subscripts 1 and 2 in Eqs. (I.10) are interchangeable. Further, the constitutive relations are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}. \quad (\text{I.11})$$

Here, the material is described by five independent constants.

If the material properties are the same in all the directions, relations (I.11) simplify to the *isotropic material* case where the number of independent constants is only two. The constitutive relations for such a material are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (c_{11} - c_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}. \quad (\text{I.12})$$

Sometimes the inverse of the material matrix in relations (I.12) is needed. One source of this matrix is Jones [2].

In engineering applications, normally the material constants are available in terms of elastic moduli, Poisson's ratios, and shear moduli. Following Jones [2], we can write the elements of stiffness matrix c_{ij} for a three-dimensional *orthotropic material* in the 1, 2, and 3

coordinate system as

$$\begin{aligned}
 c_{11} &= \frac{1 - \nu_{23}\nu_{32}}{E_2 E_3 \Delta}, & c_{22} &= \frac{1 - \nu_{13}\nu_{31}}{E_1 E_3 \Delta}, \\
 c_{33} &= \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2 \Delta}, & c_{12} &= \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2 E_3 \Delta} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1 E_3 \Delta}, \\
 c_{13} &= \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2 E_3 \Delta} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{E_1 E_3 \Delta}, & c_{23} &= \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1 E_2 \Delta} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{E_1 E_2 \Delta}, \\
 c_{44} &= G_{23}, & c_{55} &= G_{31}, & c_{66} &= G_{12},
 \end{aligned} \tag{I.13}$$

where $\Delta = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13})/(E_1 E_2 E_3)$.

For a three-dimensional isotropic material, the constitutive equations, after replacing in relations (I.12) the stiffness coefficients c_{ij} by the engineering constants E , G , and ν , can be written in terms of X , Y , and Z coordinate axes as

$$\begin{aligned}
 \sigma_{xx} &= \frac{E}{1 + \nu} \left(\frac{\nu}{1 - 2\nu} e + \epsilon_{xx} \right), \\
 \sigma_{yy} &= \frac{E}{1 + \nu} \left(\frac{\nu}{1 - 2\nu} e + \epsilon_{yy} \right), \\
 \sigma_{zz} &= \frac{E}{1 + \nu} \left(\frac{\nu}{1 - 2\nu} e + \epsilon_{zz} \right), \\
 \tau_{xz} &= G\gamma_{xz} = \frac{E}{2(1 + \nu)} \gamma_{xz}, \\
 \tau_{yz} &= G\gamma_{yz} = \frac{E}{2(1 + \nu)} \gamma_{yz}, \\
 \tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy},
 \end{aligned} \tag{I.14}$$

where $e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$.

A thin plate is essentially two-dimensional, and hence the constitutive equations reduce to those for a two-dimensional structure. Thus, for an isotropic material (X - and Y -axis coincide with principal material axes),

$$\begin{aligned}
 \sigma_{xx} &= \frac{E}{1 - \nu^2} (\epsilon_{xx} + \nu\epsilon_{yy}), & \sigma_{yy} &= \frac{E}{1 - \nu^2} (\epsilon_{yy} + \nu\epsilon_{xx}), \\
 \tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy}.
 \end{aligned} \tag{I.15}$$

In an orthotropic material, for a lamina in the LT -plane (Fig. A.1), the plane stress state is defined by setting

$$\sigma_{TT'} = 0, \quad \tau_{TT'} = 0, \quad \tau_{LT'} = 0. \tag{I.16}$$

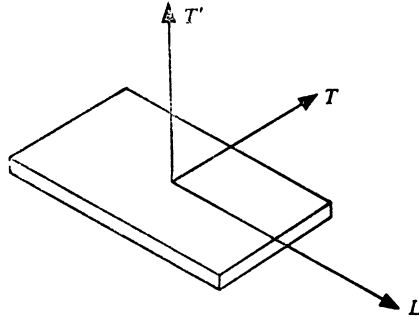


Fig. A.1 Orthotropic Lamina.

For such a case, Eqs. (I.12) can be reduced by replacing c_{ij} by Q_{ij} (only for a two-dimensional structure) to

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{Bmatrix}. \quad (\text{I.17})$$

Q_{ij} for $i, j = 1, 2, 6$ with Q_{16} and Q_{26} equal to zero are written in terms of the engineering constants E_L , E_T , ν_{LT} , G_{LT} as

$$\begin{aligned} Q_{11} &= \frac{E_L}{1 - \nu_{LT}\nu_{TL}}, & Q_{22} &= \frac{E_T}{1 - \nu_{TL}\nu_{LT}}, \\ Q_{12} &= \frac{\nu_{LT}E_T}{1 - \nu_{LT}\nu_{TL}} = \frac{\nu_{TL}E_L}{1 - \nu_{LT}\nu_{TL}}, & Q_{66} &= G_{LT} \end{aligned} \quad (\text{I.18})$$

and these engineering constants are related as

$$\nu_{LT}/E_L = \nu_{TL}/E_T. \quad (\text{I.19})$$

Equations (I.17) are the constitutive equations in the principal material directions for an orthotropic material.

In many materials, e.g., plywood where one layer of wood is placed on the other and the two layers are glued together, the orthotropic axes do not coincide with the structural axes. In such instances, the constitutive equations about the principal axes have to be rewritten about the structural axes. Suppose that the X - and Y -axis correspond to the structural axes and that the L - and T -axis correspond to the principal material axes (Fig. A.2). As shown in the figure, θ is positive. Therefore, the stresses in the transformed plane (see any standard text on mecha-

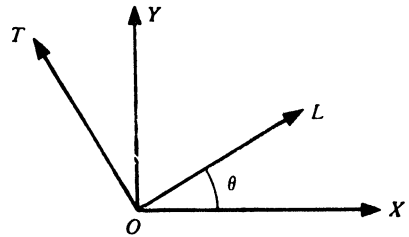


Fig. A.2 Coordinate System for Material and Structural Axes.

nics of materials) are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & -2 \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & 2 \sin \theta \cos \theta \\ \sin \theta \cos \theta & -\sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}. \quad (\text{I.20})$$

A similar relation can be written for the transformation of strains. Substituting the transformation equations (I.20) in Eqs. (I.17), we obtain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad (\text{I.21})$$

where

$$\begin{aligned} \bar{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta, \\ \bar{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta, \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12}(\sin^4 \theta + \cos^4 \theta), \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta, \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta, \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66}(\sin^4 \theta + \cos^4 \theta). \end{aligned} \quad (\text{I.22})$$

The elements of the matrix in Eqs. (I.21) are termed as transformed reduced stiffnesses. It should be noted that this matrix is full unlike the matrix in Eqs. (I.17) where there are zero elements and only four independent material constants Q_{11} , Q_{12} , Q_{22} , and Q_{66} . Since an orthotropic lamina retains its orthotropic characteristics in the principal material directions, it is referred to as a *generally orthotropic lamina*.

A composite laminate is made up of several orthotropic laminae, each having different material properties given by Eqs. (I.21).

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- [2] Jones, R.M., Mechanics of Composite Materials, Scripta Book Company, Washington, 1975.

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